Asymptotic solutions of the nonlinear Boltzmann equation for dissipative systems

Matthieu H. Ernst[1] and Ricardo Brito[2]

[1] Instituut voor Theoretische Fysica, Universiteit Utrecht, Postbus 80.195, 3508 TD Utrecht, The Netherlands. Email: ernst@phys.uu.nl
[2] GISC and Depto de Física Aplicada I, Universidad Complutense, 28040 Madrid, Spain. Email: brito@seneca.fis.ucm.es

Summary. Analytic solutions $F(v,t)$ of the nonlinear Boltzmann equation in d dimensions are studied for a new class of dissipative models, called inelastic soft spheres, interacting through pseudo-power law repulsions, and characterized by a collision frequency $\propto g^{\nu}$, where g is the relative speed of the colliding particles. These models embed inelastic hard spheres ($\nu = 1$) and inelastic Maxwell models ($\nu = 0$). The systems are either freely cooling without energy input or driven by thermostats, e.g. white noise, and approach stable nonequilibrium steady states, or marginally stable homogeneous cooling states, where the data, $v_0^d(t)F(v,t)$ plotted versus $c = v/v_0(t)$, collapse on a scaling or similarity solution $f(c)$, where $v_0(t)$ is the r.m.s. velocity. The dissipative interactions generate overpopulated high energy tails, described generically by stretched Gaussians, $f(c) \sim \exp[-\beta c^b]$ with $0 < b < 2$, where $b = \nu$ with $\nu > 0$ in free cooling , and $b = 1 + \frac{1}{2}\nu$ with $\nu \geq 0$ when driven by white noise. Power law tails, $f(c) \sim 1/c^{a+d}$, are only found in marginal cases, where the exponent a is the root of a transcendental equation. The stability threshold depend on the type of thermostat, and is for the case of free cooling located at $\nu = 0$. Moreover we analyze an inelastic BGK-type kinetic equation with an energy dependent collision frequency coupled to a thermostat, that captures all qualitative properties of the velocity distribution function in Maxwell models, as predicted by the full nonlinear Boltzmann equation, but fails for harder interactions with $\nu > 0$. This review was completed early February 2003 and covers most of the relevant literature available at that time.

1 Introduction

Classic kinetic theory [1–5] deals with elastic particles with energy conserving dynamics. The system is described by the single particle distribution function, whose time evolution is governed by the nonlinear Boltzmann equation. The asymptotic states of such systems follow the universal laws of thermodynamics, and the distribution function is the Maxwell Boltzmann distribution. This scenario does not apply to dissipative systems, where energy is lost in inelastic interactions.

In elastic systems the approach to asymptotic states is characterized by a *kinetic* stage of rapid relaxation in velocity space to a locally homogeneous equilibrium state, followed by a *hydrodynamic* stage of slow approach to a globally homogeneous equilibrium state. The time scale in the kinetic stage is the mean free time t_{mf} between collisions. In the kinetic theory of inelastic systems [6–12] the type of decay depends on the energy supply to the dissipative system. Without energy supply there is first a kinetic stage of rapid relaxation on the time scale t_{mf} to a locally homogeneous adiabatic state, the *homogeneous cooling* state, described by *scaling* or *similarity* solutions with a slowly changing parameter, at least for weakly inelastic systems. With energy supply the evolution is more similar to the elastic case with, however, equilibrium states replaced by non-equilibrium steady states. The velocity distributions in these adiabatic or steady states are very different from a Maxwell Boltzmann distribution. The subsequent stage of evolution involves transport phenomena and complex hydrodynamic phenomena of clustering and pattern formation [12, 13].

The interest in granular matter in general has strongly stimulated new developments in the kinetic theory of granular fluids and gases, which show surprising new physics. A granular fluid is a collection of small or large macroscopic particles, with short range repulsive hard core interactions, in which energy is lost in inelastic collisions, and the system cools when not driven. When rapidly driven, gravity can be neglected. The dynamics is based on binary collisions and ballistic motion between collisions, which *conserve total momentum*. So these systems can be considered to be a granular *fluid or gas*. The prototypical model for these so-called rapid granular flows is a fluid or gas of perfectly smooth mono-disperse inelastic hard spheres, and its non-equilibrium behavior can be described by the nonlinear Boltzmann equation [6–12]. The inelastic collisions are modeled by a *coefficient of restitution* α $(0 < \alpha < 1)$, where $\left(1 - \alpha^2\right)$ measures the degree of inelasticity.

This review focuses on the first stage of evolution, and studies the velocity distribution $F(v, t)$ in spatially homogeneous states of inelastic systems. For that reason most of the citations, given in this article, only refer to kinetic theory studies of $F(v, t)$, based on the nonlinear Boltzmann equation or on simplified versions of it, like the BGK- kinetic equation. We refer to the literature for further interesting developments in the theory of inelastic fluids of more complicated systems with internal degrees of freedom, unlike particles, lack of equipartition (see reviews in this volume by Luding et al. [14], and by Baldassarri et al. [15]), about Green-Kubo formulas in granular fluids (see the review by Brey et al. [16], and Goldhirsch and van Noije [17]), and the article by Cáceres [18] about anomalous velocity distributions in non-equilibrium steady states of general systems out of equilibrium, based on the Langevin and Fokker-Planck equations. The revival [19–31] in kinetic theory of inelastic systems has been strongly stimulated by the increasing sophistication of experimental techniques [32, 33], which make direct measurements of velocity distributions feasible in non-equilibrium steady states. In this review we also include inelastic generalizations [34] of the classical repulsive power law interactions [2–4], which embed both the inelastic hard spheres ($\nu = 1$), as well as the recently much studied [35–45] inelastic Maxwell models ($\nu = 0$) in a single class of models, which are characterized by a collision frequency $\propto g^{\nu}$, where g is the relative speed of the colliding particles.

In fact, the kinetic theory for such models is of interest in its own right, as the majority of inter-particle interactions in macroscopic systems involve some effects of inelasticity. Our goal is to expose the generic and universal features of the velocity distributions in dissipative fluids, and to compare them with conservative fluids to highlight the differences.

A classical problem in kinetic theory is the possibility of overpopulated high energy tails in velocity distributions [46, 47], as many physical and chemical processes only occur above a certain energy threshold. Consequently, this old problem has received a new stimulus through a large amount of recent theoretical and experimental studies on tail distributions in many particle systems with inelastic interactions. From the point of view of kinetic theory the intriguing question is, what is the generic feature causing overpopulated tails, possibly even power law tails, in systems of *inelastic* particles, how does the overpopulation depend on the scattering cross sections, and on the different forms of energy input. The generic feature is the mechanism for overpopulation, and not the specific shape of the tails.

Finally, from the point of view of nonequilibrium steady states, the structure of velocity distributions in elastic and dissipative systems, including the high energy tail, is a subject of continuing research, as the universality of the Gibbs' state of thermal equilibrium is lacking outside thermal equilibrium, and a possible classification of generic structures would be of great interest in many fields of non-equilibrium statistical mechanics.

The *plan of the paper* is as follows: in Section 2 we discuss a simple inelastic BGK- or single-relaxation-time model [48] to illustrate the phenomenon of power law tails. The exponent in the algebraic tail depends qualitatively in the same manner on the degree of inelasticity as in 2- and 3-dimensional Maxwell models. In Section 3 the nonlinear Boltzmann equation is constructed for inelastic generalizations of the classical repulsive power law potentials, to which we refer as Inelastic Soft Spheres or ISS-models. This is done for freely cooling as well as for systems driven by thermostats or heat sources. Section 4 gives a systematic analysis for the energy balance equation, it derives the nonlinear integral equation for the scaling or similarity solution, denoted by $f(c)$, and presents an asymptotic analysis of the high energy tails in the form of stretched Gaussians, $f(c) \sim \exp\left[-\beta c^{b}\right]$ where $b < 2$. The method used can only be applied to ISS-models with $\nu > 0$, where the exponent $b = b(\nu)$ is found as a simple function of ν. The case of freely cooling Maxwell models ($\nu = 0$) forms an exceptional borderline case, discussed in Section 5. Here algebraic tails, $f(c) \sim 1/c^{d+a}$, are found, where the exponent a is determined by a transcendental equation. It yields $a = a(\alpha)$ as a function of the degree of inelasticity. We end with some perspectives and conclusions.

2 Inelastic BGK Model

2.1 Kinetic equations

The goal of this section is to present in the nutshell of a simple Bhatnagar-Gross-Krook (BGK) model [48] a preview of many of the qualitative features of velocity relaxation in homogeneous systems.

A crude scenario for the relaxation without energy input suggests that the system will cool down due to inelastic collisions, and the velocity distribution $F(v,t)$ will approach a Dirac delta function $\delta(\vec{v})$ as $t \to \infty$, while the width or r.m.s. velocity $v_0(t)$ of this distribution, defined as $\langle v^2 \rangle = \frac{1}{2} d v_0^2$, is shrinking. With a constant supply of energy, the system can reach a non-equilibrium steady state (NESS).

To model this evolution we use a simple BGK-type kinetic equation, introduced in 1996 by Brey et al. [49],

$$\partial_t F(v,t) - D\nabla_{\vec{v}}^2 F(v,t) = -\omega_\nu(t)\left[F(v,t) - F_0(v,t)\right]. \tag{1}$$

We have added a heating term, $-D\nabla_{\vec{v}}^2 F(v,t)$ to the usual BGK equation which represents the heating by a white noise of strength D [3]. Here the mean collision frequency, $\omega_\nu(t) = 1/t_{mf}$, is a function of the r.m.s. velocity $v_0(t)$, chosen as $\omega_\nu = v_0^\nu$, in preparation of section 3.1. The kinetic equation describes the relaxation of $F(v,t)$ with a time-dependent rate $\omega_\nu(t)$ towards a Maxwellian with a width proportional to $\alpha v_0(t)$, defined by

$$F_0(v,t) = \left(\sqrt{\pi}\alpha v_0\right)^{-d} \exp\left[-\left(v/\alpha v_0\right)^2\right] \equiv (\alpha v_0)^{-d}\phi(c/\alpha), \tag{2}$$

where $c = v/v_0$. The constant α $(0 < \alpha < 1)$ is related to the inelasticity, $\gamma = \frac{1}{2}\left(1-\alpha^2\right)$, of the model, and the totally inelastic limit $(\alpha \to 0)$ is ill-defined in this model, as the mean energy is divergent for $\alpha = 0$. Note that the 'loss term', $-\omega_\nu F$, and the 'gain term', $+\omega_\nu F_0$, contribute respectively $-\omega_\nu v_0^2$ and $+\omega_\nu \alpha^2 v_0^2$ to the net energy loss rate in the energy balance equation.

2.2 Free Cooling ($D = 0$)

The cooling law of the mean square velocity $\langle v^2 \rangle = \frac{1}{2} d v_0^2$, or the granular temperature $T \propto v_0^2$, is obtained by applying $\int d\vec{v} v^2 (\dots)$ to (1) with $\omega_\nu = v_0^\nu$. The result is $\dot{v}_0 = -\gamma v_0^{\nu+1}$, yielding

$$v_0(t) = v_0(0) / \left[1 + \nu\gamma t v_0^\nu(0)\right]^{1/\nu}. \tag{3}$$

The result is a homogeneous cooling law, $T \sim t^{-2/\nu}$, which agrees with Haff's law [9] for $\nu = 1$, corresponding to inelastic hard spheres. Note that for negative ν the homogeneous cooling law takes the form,

[3] For a systematic discussion of driven systems, see Section 3.3

$$v_0(t) = v_0(0)\left[1 - |\nu|\gamma t / v_0^{|\nu|}(0)\right]^{1/|\nu|}, \tag{4}$$

i.e. at $t = t_s \equiv v_0^{|\nu|}/|\nu|\gamma$ the r.m.s. velocity and the mean kinetic energy are vanishing, which is *unphysical*, and so are the BGK-models with $\nu < 0$.

As indicated in the introduction, the long time behavior of $F(v,t)$ in free cooling is determined by a scaling or similarity solution of the form $F(v,t) = v_0^{-d}(t)\, f(v/v_0(t))$. We insert this ansatz in (1), eliminate $\dot{v}_0 = -\gamma v_0^{\nu+1}$, and obtain the scaling equation,

$$c\frac{d}{dc}f + (d+a)\,f = \frac{a}{\alpha^d}\phi\left(\frac{c}{\alpha}\right), \tag{5}$$

where a is defined as,

$$a = 1/\gamma = 2/\left(1-\alpha^2\right). \tag{6}$$

We also note that the scaling equation is independent of ν. The exact solution of this equation is:

$$f(c) = \frac{A}{c^{d+a}} + \frac{a}{\alpha^d \pi^{d/2}}\left(\frac{1}{c^{d+a}}\right)\int_0^c du\, u^{d+a-1}\exp\left[-u^2/\alpha^2\right]. \tag{7}$$

The integration constant A is fixed by the normalizations,

$$\int d\vec{c}\{1, c^2\} f(c) = \{1, \tfrac{1}{2}d\}, \tag{8}$$

where $d\vec{c} = \Omega_d c^{d-1} dc$ with $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ being the surface area of a $d-$dimensional hyper-sphere. The normalization integral converges only near $c \simeq 0$ if $A = 0$. Then the solution (7) with $A = 0$ is identical to the scaling form, obtained in [49] for $\nu = 1$.

For velocities far above thermal, i.e. $c \gg 1$, the solution has a power law tail,

$$f(c) \sim \frac{a\alpha^a}{\pi^{d/2}}\left(\frac{1}{c^{d+a}}\right)\int_0^\infty du\, u^{d+a-1}e^{-u^2} = \frac{a\alpha^a \Gamma\left(\frac{d+a}{2}\right)}{2\pi^{d/2}}\left(\frac{1}{c^{d+a}}\right), \tag{9}$$

with exponent $a = 1/\gamma = 2/\left(1-\alpha^2\right)$, such that $\langle c^2 \rangle$ is bounded for $\alpha > 0$. The exact solution (7), including its high energy tail, is independent of the exponent ν, that determines the energy dependence of the mean collision frequency $\omega_\nu = v_0^\nu$ in the BGK model.

As we shall see in Section 5, a similar heavily overpopulated tail, $f(c) \sim 1/c^{d+a}$ with $d > 1$, will also be found in freely cooling Maxwell model with $\omega_0 = 1$ $(\nu = 0)$. There the exponent $a(\alpha)$ takes in the elastic limit $(\alpha \to 1)$ the form $a \simeq 1/\gamma_0 = 4d/\left(1-\alpha^2\right)$. However, in the *general class* of ISS-models with collision frequency $\omega_\nu \sim v_0^\nu$ with $\nu > 0$ the tails are *not* given by *power laws*, but by *stretched Gaussians*, $f(c) \sim \exp\left[-\beta c^b\right]$ with $0 < b = b(\nu) < 2$. These models will be introduced and discussed in Section 3.2.

2.3 NESS ($D \neq 0$)

Next we extend the result of [49] to an inelastic BGK-model with $\nu \geq 0$, driven by white noise. By applying $\int d\vec{v}\, v^2$ to (1) we obtain the temperature balance equation at *stationarity*,

$$\frac{dv_0^2}{dt} = 4D - 2\gamma v_0^{\nu+2} = 0, \tag{10}$$

where the collisional dissipation, $2\gamma v_0^{\nu+2}$, is compensated by energy input from the external white noise. Also note that for $\nu < -2$ the fixed point solution $v_0(\infty)$ in (10) still exists, but it is unstable [34]. If $v_0(0) < v_0(\infty)$, then $v_0(t)$ vanishes as $t \to \infty$, and if $v_0(0) > v_0(\infty)$ then $v_0(t)$ diverges [4].

To obtain the solution $F(v,\infty)$ of (1) in the NESS we rescale $F(v,\infty) = v_0^{-d}(\infty) f(v/v_0(\infty))$ to the standard width $\langle c^2 \rangle = \frac{1}{2}d$, substitute the rescaled form in the kinetic equation, and eliminate D, using the stationarity condition (10) as well as (6). This yields the rescaled equation in *universal* form,

$$\frac{1}{c^{d-1}} \frac{d}{dc} c^{d-1} \frac{d}{dc} f(c) - 2af(c) = -\frac{2a}{\alpha^d} \phi\left(\frac{c}{\alpha}\right), \tag{11}$$

where the normalizations (8) are imposed. The O.D.E. shows that $f(c)$ is independent of the noise strength, D, and does not contain any dependence on the initial data. This equation can be solved exactly, and more details will be published in [34]. However, for the purpose of this section, we only want to extract from the differential equation (11) the asymptotic form of $f(c)$. In that case we may neglect in (11) the inhomogeneity $\phi(c/\alpha) \sim \exp\left[-(c/\alpha)^2\right]$ for $c \gg \alpha$, and find the asymptotic solution for the BGK-model driven by white noise, i.e.

$$\begin{aligned} &f(c) \sim \exp\left[-\beta c^b\right] \\ &b = 1;\ \beta = \sqrt{2a} = 2/\sqrt{1-\alpha^2}. \end{aligned} \tag{12}$$

The constant β is independent of the parameter ν. The exponentially decaying high energy tail is also a 'stretched' Gaussian, which is overpopulated when compared to a Maxwellian, but the overpopulation is much less heavy than is the freely cooling case (9) with an algebraic tail.

As we shall see in Section 4.3, a similar exponential high energy tail will be found in the white noise driven Maxwell model ($\nu = 0$; $b = 1$; $\beta = \sqrt{8/(1-\alpha^2)}$), but not in the general class of ISS-models, where the stretching exponent b takes a value in the interval $0 < b = \overline{b}(\nu) \leq 2$.

[4] Thanks are due to E. Trizac and A. Barrat for pointing out to us that the energy balance equations in this article have stability thresholds that are $\nu-$dependent.

3 Basics of inelastic scattering models

3.1 Boltzmann equation as a stochastic process

The nonlinear Boltzmann equation for dissipative interactions in the *homogeneous cooling state* can be put in a broader perspective, that covers both elastic and inelastic collisions, as well as interactions where the scattering of particles is described either by conservative (deterministic) forces, or by stochastic ones. To do so it is convenient to interpret the Boltzmann equation as a stochastic process, similar to the presentations in the classical articles of Waldmann [1], and Uhlenbeck and Ford [5], or in Ulam's stochastic model [50] showing the basics of the approach of a one-dimensional gas of elastic particles towards a Maxwellian distribution.

Consider a spatially homogeneous fluid of elastic or inelastic particles in $d-$dimensions, specified by their velocities $(\vec{v}, \vec{w}, \cdots)$, and interacting through binary collisions, $(\vec{v}, \vec{w}) \rightarrow (\vec{v}\,', \vec{w}')$, that are described by transition probabilities. To describe fluids out of equilibrium the total momentum $\vec{G} = \frac{1}{2}(\vec{v}+\vec{w})$ needs to be conserved in a binary collision. The outgoing or post-collision velocities $(\vec{v}\,', \vec{w}')$ can be parametrized in terms of the incoming velocities $(\vec{v}, \vec{w})$, and an impact (unit) vector $\vec{n}$, that is chosen on the surface of a unit sphere with a certain probability, proportional to the collision frequency $\mathrm{A}(\vec{g}, \vec{n})$, that in general depends on the relative speed $g = |\vec{v} - \vec{w}|$ of the colliding particles, and the angle between $\hat{\vec{g}}$ and $\vec{n}$, where $\hat{\vec{a}}$ denotes a unit vector.

In this article we consider the simplest case of modeling the inelastic collisions through a velocity-independent coefficient of restitution α $(0 < \alpha < 1)$, where the component $\mathrm{g}_{\|} \equiv \vec{g} \cdot \vec{n}$ is not only reflected, as in elastic collisions, but also reduced in size by a factor α, i.e.

$$\mathrm{g}_{\|}{}' = -\alpha \mathrm{g}_{\|}. \tag{13}$$

The components $\vec{g}_{\perp} = \vec{g} - \mathrm{g}_{\|}\vec{n}$, orthogonal to $\vec{n}$, remain unchanged. More explicitly, with the help of momentum conservation we find for the post-collision velocities resulting from the direct collisions $(\vec{v}, \vec{w}) \rightarrow (\vec{v}\,', \vec{w}') = (\vec{v}\,^*, \vec{w}^*)$,

$$\begin{aligned} \vec{v}\,^* &= \vec{v} - \tfrac{1}{2}(1+\alpha)\,\vec{g}\cdot\vec{n}\vec{n} \\ \vec{w}^* &= \vec{w} + \tfrac{1}{2}(1+\alpha)\,\vec{g}\cdot\vec{n}\vec{n}. \end{aligned} \tag{14}$$

The corresponding energy loss in such a collision is,

$$\Delta E = \tfrac{1}{2}\left(v^{*2} + w^{*2} - v^2 - w^2\right) = -\tfrac{1}{4}\left(1-\alpha^2\right)(\vec{g}\cdot\vec{n})^2\,, \tag{15}$$

where $\left(1-\alpha^2\right)$ measures the degree of inelasticity. The value $\alpha = 1$ describes elastic collisions. In the special case of hard spheres the impact vector $\vec{n}$ is the unit vector along the line of centers of the colliding spheres at contact, as illustrated in Figure 1.

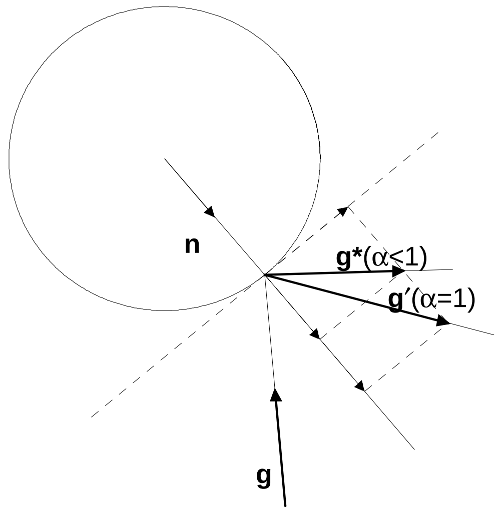

Fig. 1. Elastic ($\alpha = 1$) and inelastic ($\alpha < 1$) scattering of hard spheres, where α is the coefficient of restitution, and the parallel component $g_\parallel$ is reflected as in (13). In elastic collisions the scattering angle $\chi = \cos^{-1}\left(\hat{\vec{g}} \cdot \hat{\vec{g}}'\right) = \pi - 2\phi$ with $\phi = \cos^{-1}\left(\hat{\vec{g}} \cdot \vec{n}\right)$ and in inelastic ones $\chi = \cos^{-1}\left(\hat{\vec{g}} \cdot \hat{\vec{g}}^*\right) = \pi - \phi - \phi^*$ with $\phi^* = \cos^{-1}\left(\hat{\vec{g}}^* \cdot \vec{n}\right)$.

The system above can be described by an isotropic velocity distribution, $F(v,t) = F(|\vec{v}|, t)$, as long as one only considers isotropic initial distributions. Its time evolution is given by the Boltzmann equation,

$$\partial_t F(v,t) = I(v|F) \equiv \int d\vec{w} d\vec{v}' d\vec{w}' \int d\vec{n} \left[W(\vec{v}, \vec{w}|\vec{v}', \vec{w}'; \vec{n}) \right. \\ \left. \times F(v',t) F(w',t) - W(\vec{v}', \vec{w}'|\vec{v}, \vec{w}; \vec{n}) F(v,t) F(w,t) \right]. \quad (16)$$

Here $W(\vec{v}', \vec{w}'|\vec{v}, \vec{w}; \vec{n})$ is the transition probability per unit time that the incoming pair state $(\vec{v}, \vec{w})$ at impact vector $\vec{n}$ is scattered into the outgoing pair state $(\vec{v}', \vec{w}')$. The loss term is the sum over all parameters of the direct collisions $(\vec{v}, \vec{w}) \rightarrow (\vec{v}', \vec{w}')$ at fixed $\vec{v}$, including a sum over impact vectors $\vec{n}$. Similarly the gain term is the sum over all parameters of the restituting collisions, $(\vec{v}', \vec{w}') \rightarrow (\vec{v}, \vec{w})$ at fixed $\vec{v}$. The transition probability for the scattering event $(\vec{v}, \vec{w}) \rightarrow (\vec{v}', \vec{w}')$, obeying the inelastic reflection law (13) and momentum conservation, is in general proportional to the collision frequency $\mathrm{A}(\vec{g}, \vec{n})$, and contains delta functions, selecting the allowed collisions,

$$W(\vec{v}', \vec{w}'|\vec{v}, \vec{w}; \vec{n}) = \mathrm{A}(\vec{g}, \vec{n})\, \delta^{(d)}\left(\vec{G}' - \vec{G}\right) \delta^{(d-1)}\left(\vec{g}'_\perp - \vec{g}_\perp\right) \delta\left(g_\parallel' + \alpha g_\parallel\right). \quad (17)$$

For conservative interactions the total energy is conserved, as well as total momentum and total number of particles. Moreover (17) shows that the transition probability is symmetric for $\alpha = 1$,

$$W\left(\vec{v}\,', \vec{w}'|\vec{v}, \vec{w}; \vec{n}\right) = W\left(\vec{v}, \vec{w}|\vec{v}\,', \vec{w}'; \vec{n}\right). \tag{18}$$

This means that the transition probabilities for elastic binary collisions satisfy the condition of *detailed balance*. Energy conservation and the detailed balance relation in combination with the $H-$theorem guarantee that the Maxwellian velocity distribution is approached at large times.

Once the detailed balance relation is obeyed, it is trivial to prove the $H-$theorem. Defining the $H-$function or entropy $S\left(t\right) = -H\left(t\right)$ as,

$$H\left(t\right) = \int d\vec{v} F\left(\vec{v}, t\right) \ln F\left(\vec{v}, t\right), \tag{19}$$

we obtain its decay rate by applying $\int d\vec{v} \ln F\left(\vec{v}, t\right)$ to (16), then symmetrizing over $\vec{v} \leftrightarrow \vec{w}$ and $\vec{v}\,' \leftrightarrow \vec{w}'$, and subsequently over $\left(\vec{v}, \vec{w}\right) \leftrightarrow \left(\vec{v}\,', \vec{w}'\right)$. The result is,

$$\partial_t H\left(t\right) = \tfrac{1}{4} \int d\vec{v} d\vec{w} d\vec{v}\,' d\vec{w}' \int d\vec{n} W\left[Y - X\right] \ln\left(X/Y\right) \leq 0, \tag{20}$$

where $X = F\left(v\right) F\left(w\right)$ and $Y = F\left(v'\right) F\left(w'\right)$ and the inequality follows from $\left[Y - X\right] \ln\left(X/Y\right) \leq 0$. The equality sign holds if and only if $X = Y$. This implies that H decreases monotonically and becomes stationary only when F approaches the Maxwellian.

The reason for reviewing these ’obvious’ properties in elastic systems is that several of them, such as energy conservation, the detailed balance relation, the $H-$theorem, and the approach to a stationary Maxwellian velocity distribution no longer hold in dissipative systems ($\alpha < 1$), as we shall see.

3.2 Boltzmann equation in standard form

The standard form of the collision term $I\left(v|F\right)$ in the Boltzmann equation for *elastic* interactions ($\alpha = 1$) contains the differential scattering cross-section $\sigma\left(g, \chi\right)$. It can be calculated from the pair potential $V\left(r\right)$ [4], and depends on the relative speed $g = |\vec{g}|$ and the scattering angle $\chi = \pi - 2\varphi$, where χ and φ are defined in Figure 1. The collision frequency is then given by $\text{A}\left(\vec{g}, \vec{n}\right) = g\sigma\left(g, \chi\right)$. For repulsive power law potentials $V\left(r\right) \sim r^{-s}$ the collision frequency is $\text{A}\left(\vec{g}, \vec{n}\right) = g^{\nu} \text{A}\left(\hat{\vec{g}} \cdot \vec{n}\right)$ with $\nu = 1 - 2\left(d - 1\right)/s$ in $d-$dimensions [51]. By definition Maxwell molecules have a collision frequency, which is independent of g, corresponding to $\nu = 0$ or $s = 2\left(d - 1\right)$. For hard spheres, $s \to \infty$ or $\nu = 1$ with $\text{A}\left(\vec{g}, \vec{n}\right) = |\vec{g} \cdot \vec{n}|\, \sigma^{d-1} = \left|\text{g}_{\|}\right| \sigma^{d-1}$, and σ is the hard sphere diameter. Using these relations the collision term in (16) can be reduced to the classical Boltzmann equation for elastic energy-conserving collisions. For *inelastic* hard spheres the collision rate is again given by $\text{A}\left(\vec{g}, \vec{n}\right) = \left|\text{g}_{\|}\right| \sigma^{d-1}$. In general, a given positive function $\text{A}\left(\vec{g}, \vec{n}\right)$ defines a stochastic scattering model, and in particular the choice, $\text{A}(\vec{g},\vec{n}) = \text{A}\left(\hat{\vec{g}}\cdot\vec{n}\right)$, defines inelastic Maxwell models [29–31]. As a simple realization of a dissipative scattering model we consider the ISS-models with a collision frequency $\text{A}(\vec{g},\vec{n}) = \text{A}_0 |\vec{g}\cdot\vec{n}|^{\nu}$ with $\nu \geq 0$. This class includes inelastic hard spheres ($\nu = 1$) and inelastic Maxwell models ($\nu = 0$). In the remainder of this article we restrict ourselves to this class of models.

To reduce the Boltzmann equation to its standard form we use the transition probabilities for dissipative interactions (17) with $\text{A}(\vec{g},\vec{n}) = \text{A}_0 \left|\text{g}_{\|}\right|^{\nu}$. Consider first the loss term in (16), insert the relation $d\vec{v}\,'d\vec{w}' = d\vec{g}_{\perp}'d\text{g}_{\|}'d\vec{G}'$, and carry out the integrations over the delta functions. The result is,

$$(\partial_t F(v))_{\text{loss}} = -\int d\vec{w}\int d\vec{n}\text{A}_0\left|\text{g}_{\|}\right|^{\nu} F(w)\,F(v)\,. \tag{21}$$

The gain term involves the transition probability for the restituting collisions $(\vec{v}\,',\vec{w}') \to (\vec{v},\vec{w})$, obtained from (17) by interchanging primed and unprimed velocities,

$$\begin{aligned}(\partial_t F(v))_{\text{gain}} =& \int d\vec{w}d\vec{g}_{\perp}'d\text{g}_{\|}'d\vec{G}'\int d\vec{n}\text{A}_0\left|\text{g}_{\|}'\right|^{\nu}\delta^{(d)}\left(\vec{G}-\vec{G}'\right)\\ &\times\ \delta^{(d-1)}\left(\vec{g}_{\perp}-\vec{g}_{\perp}'\right)\frac{1}{\alpha}\delta\left(\text{g}_{\|}'+\frac{1}{\alpha}\text{g}_{\|}\right)F(w')\,F(v')\\ =&\int d\vec{w}\int d\vec{n}\,(1/\alpha)\,\text{A}_0\left|\text{g}_{\|}/\alpha\right|^{\nu}F(w^{**})\,F(v^{**})\,.\end{aligned} \tag{22}$$

In the second integral we have carried out the integrations over the primed velocities, and used the following relations for the restituting velocities,

$$\begin{aligned}\vec{v}\,' =&\vec{G}'+\tfrac{1}{2}\vec{g}_{\perp}'+\tfrac{1}{2}\text{g}_{\|}'\vec{n} = \vec{G}+\tfrac{1}{2}\vec{g}_{\perp}-\tfrac{1}{2\alpha}\text{g}_{\|}\vec{n}\\ =&\vec{v}-\tfrac{1}{2}\left(1+\tfrac{1}{\alpha}\right)\vec{g}\cdot\vec{n}\vec{n}\equiv\vec{v}^{\,**}\\ \vec{w}' =&\vec{w}+\tfrac{1}{2}\left(1+\tfrac{1}{\alpha}\right)\vec{g}\cdot\vec{n}\vec{n}\equiv\vec{w}^{**}.\end{aligned} \tag{23}$$

In the first equality $\vec{v}\,'$ has been expressed in center of mass and relative velocities. In the second equality we have used the inelastic collision law (13) and conservation of total momentum, and the very last equality defines the restituting velocities, $(\vec{v}^{\,**},\vec{w}^{**})$. They are the incoming velocities that result in the scattered velocities $(\vec{v},\vec{w})$, described by the inverse of the transformation (14).

The space-homogeneous Boltzmann equation in its standard form is then obtained by combining (21) and (22) with $\text{A}(\vec{g},\vec{n}) = |\vec{g}\cdot\vec{n}|^{\nu}$ to yield,

$$\begin{aligned}\partial_t F(v) =&I(v|F)\\ I(v|F) =& \int_{\vec{n}}\int d\vec{w}\,|\vec{g}\cdot\vec{n}|^{\nu}\left[\frac{1}{\alpha^{\nu+1}}F(v^{**})\,F(w^{**})-F(v)\,F(w)\right],\end{aligned} \tag{24}$$

where $\int_{\vec{n}}(\cdots) = (1/\Omega_d)\int d\vec{n}\,(\cdots)$ is an average over a $d-$dimensional unit sphere, and we have absorbed constant factors in the time scale. Here $\nu = 1$ corresponds to inelastic hard spheres and $\nu = 0$ to inelastic Maxwell models. Velocities and time have been dimensionalized in terms of the width and the mean free time of the initial distribution. Moreover, the Boltzmann equation obeys conservation of particle number and total momentum, but the average kinetic energy or granular temperature, $T \propto \langle v^2 \rangle$, decreases in time on account of the dissipative collisions, i.e.

$$\int d\vec{v}\left(1,\ \vec{v},\ v^2\right) F\left(v,t\right) = \left(1,\ 0,\ \tfrac{1}{2} d v_0^2\left(t\right)\right), \tag{25}$$

where $v_0\left(t\right)$ is the r.m.s. velocity. The inelastic scattering models with collision frequency $\mathrm{A} \sim g^{\nu}$ $(\nu > 0)$, are the inelastic analogs of the deterministic models with repulsive power law potentials, $V\left(r\right) \sim r^{-s}$ with $2\left(d-1\right) < s < \infty$, introduced at the start of this section. The inelastic case with $\nu = 2$ corresponds to an exactly solvable stochastic scattering model with energy conservation, known as Very Hard Particle model [51].

The most basic and most frequently used model for dissipative systems with short range hard core repulsion is the Enskog-Boltzmann equation for inelastic hard spheres in $d-$dimensions [6], which simplifies in the spatially homogeneous case to (24) with $\nu = 1$. Recently, inelastic Maxwell models have been studied extensively. Ben-Naim and Krapivsky [29] introduced the one-dimensional version of (24) with $(d = 1, \nu = 0)$, and Bobylev et al [30] have introduced a three-dimensional Maxwell model with $\mathrm{A}\left(\vec{g},\vec{n}\right) = \mathrm{A}_0 \left|\hat{\vec{g}}\cdot\vec{n}\right|$. The Maxwell models in (24) for general d were first considered in [37, 39]. We frequently use the terms 'harder' (larger ν) and 'softer' interactions (smaller ν), which only refers to the energy dependence of the collision frequency, $\mathrm{A} \propto g^{\nu}$, at large relative speed g. We only use a relative criterion for 'harder' and 'softer', and not an absolute one. Moreover the faster the collision frequency increases at large impact velocities, the more rapidly the high energy tail of $F\left(v,t\right)$ relaxes relative to the bulk values of $F\left(v,t\right)$ with v in the thermal range.

3.3 Cooling and driven systems

Homogeneous cooling and scaling:
An inelastic fluid without energy input will cool down due to the collisional dissipation in (15). In experimental studies of granular fluids energy has to be supplied at a constant rate to keep the system in a non-equilibrium steady state, while in analytic and simulation studies freely cooling systems can be studied directly. Without energy input the velocity distribution $F\left(v,t\right)$ will approach a Dirac delta function $\delta\left(\vec{v}\right)$ as $t \to \infty$, and all moments approach zero, including the width $v_0\left(t\right)$.

However, an interesting structure is revealed when velocities, $\vec{c} = \vec{v}/v_0\left(t\right)$, are measured in units of the instantaneous width $v_0\left(t\right)$, and the long time limit is taken while keeping $\vec{c}$ constant, the so-called *scaling limit*. Monte Carlo simulations [35] of the Boltzmann equation suggest that in this limit the rescaled velocity distribution of the homogeneous cooling state can be collapsed on a scaling form or similarity solution $f(c)$. These observations seem to indicate that the long time behavior of $F(v,t)$ in freely cooling systems approaches a simple, and to some extent *universal*, form $f(c)$, which is the same for different initial distributions. Such *scaling* or *similarity* solutions have the structure,

$$F(v,t) = (v_0(t))^{-d} f(v/v_0(t)), \tag{26}$$

where $\vec{c} = \vec{v}/v_0(t)$ is the scaling argument. Then $f(c)$ satisfies the normalizations,

$$\int d\vec{c} f(c) = 1; \quad \int d\vec{c} c^2 f(c) = \tfrac{1}{2} d, \tag{27}$$

on account of (25). Substitution of the scaling ansatz (26) in the Boltzmann equation (24) then leads to a *separation of variables*, i.e.

$$\begin{aligned} I(c|f) =& \gamma (df + cdf/dc) = \gamma \vec{\nabla}_{\vec{c}} \cdot \vec{c} f \\ \dot{v}_0 =& -\gamma v_0^{\nu+1}, \end{aligned} \tag{28}$$

where γ is a separation constant.

The question is then: Can one determine physically acceptable solutions? Here a short history. The first kinetic model with dissipative interactions, that has been solved exactly for the case of free cooling, *and* exhibits a heavily overpopulated high energy tail, $f(c) \sim 1/c^{d+a}$, is the inelastic BGK model, discussed in Section 2. Asymptotic solutions ($c \gg 1$) of the scaling equation for inelastic hard spheres [21, 22] have predicted the existence of exponential *high energy tails*, $f(c) \sim \exp[-\beta c]$. We talk about *tails*, *over-populations* or stretched Gaussians in $f(c)$, when the ratio of $f(c)$ and a Gaussian is an increasing function of c at $c \gg 1$. The predictions about high energy tails were later confirmed in great detail by Monte Carlo simulations of the long time solutions of the nonlinear Boltzmann equation for inelastic hard spheres [25, 26], as well as in further analytic work [23].

Similarity solutions for freely cooling inelastic Maxwell models [29, 30] where first studied in terms of the scaling variable, $(1-\alpha^2)t$, relating large times and small inelasticities [30], but the solutions obtained turned out to be unphysical, i.e. *non-positive*. The first exact positive similarity solution was found by Baldassarri et al. [35] for the one-dimensional Maxwell model. It shows a surprisingly strong high energy tail of algebraic type, $\sim 1/c^4$. Using Monte Carlo simulations of the Boltzmann equation, these authors also showed that rather general initial distributions approach towards this scaling form for long times. Algebraic tails, $f(c) \sim 1/c^{d+a}$, for Maxwell models in higher dimensions have been obtained analytically in [37–41], and the actual approach in time towards such scaling forms has been studied analytically in [41, 42] for general classes of initial distributions.

Driving and steady states:
Heating may be described by applying an external stochastic force to the particles in the system, or by connecting the system to a thermostat, which may be modeled by a frictional force. For example, the friction force $\gamma\vec{v}$ – here with a negative friction coefficient $(-\gamma)$ – is called a Gaussian thermostat. Complex fluids (e.g. granular) subject to such forces can be described by the microscopic equations of motion for the particles, $\dot{\vec{r}}_i = \vec{v}_i$, and $\dot{\vec{v}}_i = \vec{a}_i + \tilde{\vec{\xi}}_i$ $(i = 1, 2, \cdots)$, where $\vec{a}_i$ and $\tilde{\vec{\xi}}_i$ are respectively the systematic and random forces per unit mass. Here $\vec{a}_i$ contains frictional forces, which may depend on velocity, and in the present case $\tilde{\vec{\xi}}_i$ represents external noise (modeling energy input), which is taken to be Gaussian white noise with zero mean, and variance,

$$\overline{\tilde{\xi}_{i,\alpha}(t)\,\tilde{\xi}_{j,\beta}(t')} = 2D\delta_{ij}\delta_{\alpha\beta}\delta(t-t'), \tag{29}$$

where α, β denote Cartesian components, and D is the noise strength. The Boltzmann equation for system driven in this manner takes the form,

$$\begin{aligned} &\partial_t F(\vec{v}) + \mathcal{F}F(\vec{v}) = I(v|F) \\ &\mathcal{F} = \vec{\nabla}_{\vec{v}} \cdot \vec{a} - D\nabla^2_{\vec{v}}. \end{aligned} \tag{30}$$

A detailed derivation on how to include frictional and stochastic forces in kinetic equations can be found, for instance, in [19, 22, 52].

When energy is supplied at a constant rate, driven systems can reach a NESS. Again there is the question about universality of these asymptotic states as $t \to \infty$. Does the NESS depend on the inelasticity, on the type of thermostat, and on the initial distribution? The basic idea to show universality is always essentially the same. Rescale the velocity distribution $F(v, \infty)$ by measuring velocities in terms of their typical size, i.e. the width $v_0(\infty)$,

$$F(v, \infty) = (v_0(\infty))^{-d} f(v/v_0(\infty)), \tag{31}$$

and analyze the scaling form $f(c)$. The scaling equation for the NESS function $f(c)$ will be analyzed in Sections 4.2 and 4.3. Here we make some comments about what is known. The scaling solutions for the NESS show again overpopulated tails in the form of stretched Gaussians, albeit with a larger stretching exponent b than in free cooling. The scaling solution for inelastic hard spheres driven by white noise was first analyzed in [22], predicting $f(c) \sim \exp\left[-\beta c^b\right]$ with $b = 3/2$. The experiments of Rouyer and Menon [32, 33], and of Aranson and Olafsen [33] seem to confirm the stretched Gaussian behavior with $b = 3/2$. Maxwell models, driven by white noise, were first studied in [29, 31], but did not give any predictions about overpopulated high energy tails. The high energy tails for Maxwell models, driven by white noise, were predicted in [40, 42] to have exponential tails, $f(c) \sim \exp[-\beta c]$. A number of more detailed analytical and numerical studies about this driven Maxwell model in one dimension [43–45, 53] have appeared as well. Moreover, as will be shown in Section 4.2, the integral equation for the scaling form $f(c)$ in free cooling is identical to the integral equation for the NESS distribution for a special thermostat, provided $F(v,\infty)$ is also rescaled to the same width as in (27). How to determine the scaling form $f(c)$ in free cooling and NESS will be described in subsequent sections.

3.4 Qualitative analysis

In order to illustrate the rich behavior of the inelastic systems, we start with the Boltzmann equation for the one-dimensional inelastic Maxwell model (24) $(d=1,\nu=0)$, where $F(v,t)$ satisfies,

$$\partial_t F(v) - D\nabla_v^2 F(v) = I(v|F), \tag{32}$$

and the collision term has the form,

$$\begin{aligned} I(v|F) &= \int dw \left[\frac{1}{\alpha} F(v^{**}) F(w^{**}) - F(v) F(w)\right] \\ &= -F(v) + \frac{1}{p}\int du F(u) F\left(\frac{v-qu}{p}\right). \end{aligned} \tag{33}$$

All velocity integrations extend over the interval $(-\infty,+\infty)$. Here the outgoing velocities (v,w), and the incoming ones (v^{**},w^{**}) are according to (23) related by,

$$v = qv^{**} + pw^{**}; \qquad w = pv^{**} + qw^{**} \tag{34}$$

with $p = 1-q = \frac{1}{2}(1+\alpha)$. By changing integration variables $w \to v^{**} = u$ with $dw = (\alpha/p)\,du$, and using the relation $w^{**} = (v-qu)/p$ one obtains the second equality in (33). The normalization of mass and mean square velocity $\langle v^2\rangle = \frac{1}{2}v_0^2$ are given by (25) for $d=1$. The temperature balance equation is obtained from (32) as,

$$\frac{dv_0^2}{dt} = 4D - 2pqv_0^2, \tag{35}$$

and describes the approach to the non-equilibrium steady state (NESS) with a width $v_0^2(\infty) = 2D/pq$, where the heating through random forces, $\sim D$, is balanced by the collisional losses.

To understand the physical processes involved we first discuss in a qualitative way the relevant limiting cases. Without the heating term $(D=0)$, equation (32) reduces to the freely cooling inelastic Maxwell model.

If one takes in addition the elastic limit ($\alpha \to 1$ or $q \to 0$), the collision laws reduce in the *one-dimensional* case to $v^{**} = w, w^{**} = v$, i.e. an exchange of particle labels; the collision term vanishes identically; every $F(v,t) = F(v)$ is a solution; there is no randomization or relaxation of the velocity distribution through collisions, and the model becomes trivial at the Boltzmann level of description, whereas the distribution function in the presence of *infinitesimal* dissipation ($\alpha \to 1$) approaches a Maxwellian.

If we turn on the noise ($D \neq 0$) at vanishing dissipation ($q = 0$), the collision term in (32) vanishes, and the granular temperature follows from (35) as $v_0^2(t) = v_0^2(0) + 4Dt$, and increases linearly with time. With stochastic heating *and* dissipation (even in infinitesimal amounts) the system reaches a NESS.

3.5 Comments

Spatial dependence:
When *spatial* dependence of the distribution function $F(\vec{r}, \vec{v}; t)$ is relevant, the collision term in the Enskog-Boltzmann equation must be slightly modified, namely the angular integration $\int_{\vec{n}}$ over the full solid angle should be replaced by $2\int_{\vec{n}} \theta(-\vec{g}\cdot\vec{n})$, where the unit step function $\theta(x)$ restricts the $\vec{n}-$integration to the pre-collision hemisphere with $\vec{g}\cdot\vec{n} < 0$ (see Figure 1). In the spatially uniform case both representations are identical because the restituting velocities (23) are even functions of $\vec{n}$.

Origin of Maxwell models:
In several papers [30, 37, 41, 43] inelastic Maxwell models have been introduced as more or less *ad hoc mathematical simplifications* of the nonlinear collision term for inelastic hard spheres. This has been done by replacing the relative velocity g in the hard sphere collision frequency $\mathrm{A}(\vec{g}, \vec{n}) = |\vec{g}\cdot\vec{n}|$ by its mean value, $\langle g\rangle\, \mathrm{A}(\vec{n}) \sim v_0(t)\, \mathrm{A}(\vec{n})$, where $v_0(t)$ is the root mean square velocity.

This procedure guarantees that the homogeneous cooling law for inelastic Maxwell model constructed in this way, is identical to the one for inelastic hard spheres, and given by Haff's law [9], $T(t) \sim t^{-2}$. The construction of inelastic Maxwell and ISS models, followed in the present article, is more in the spirit of [29], i.e. by defining the collision term through transition probabilities for the scattering process $(\vec{v}, \vec{w}) \leftrightarrow (\vec{v}\,', \vec{w}')$ with the proper constraints.

Violation of $H-$theorem:
As the detailed balance symmetry and the $H-$theorem are lacking in dissipative interaction models, there is no guarantee that the entropy $S(t) = -H(t)$ is *non-decreasing*. In fact the solutions of the Boltzmann equation for such dissipative interaction models approach [41] for long times to a scaling form, defined in (26). By inserting such solutions into (19) and anticipating the cooling law (3) in the next section, one easily verifies that the entropy in the *scaling* state keeps *decreasing* as t becomes large, i.e.

$$S(t) = -H(t) \sim -(d/\nu)\ln t. \tag{36}$$

This result is typical for pattern forming mechanisms in configuration space, where spatial order or correlations are building up, as well as in chaos theory, where the rate of irreversible entropy production is negative on an attractor

[54]. Moreover, there is no fundamental objection against decreasing entropies in an open system in contact with a reservoir, which is here the energy sink formed by the dissipative collisions. The dynamics in $N-$particle velocity space corresponds to a contracting flow $d\vec{v}^* d\vec{w}^* = \alpha d\vec{v} d\vec{w}$ where $\alpha < 1$, where the probability is contracting onto an attractor. This is a well known phenomenon in chaos theory.

Particles with pseudo-power law repulsion:
A system of N inelastic hard spheres is a model with microscopic particles, characterized by positions and velocities, and interacting via well-defined force laws, that can be studied by means of MD simulations. However microscopic particles with dissipative inter-particle forces, such as inelastic Maxwell molecules and ISS-models, are stochastic models, only defined in $N-$particle velocity space. Molecular Dynamics simulations can not be performed for such models.

Modeling dissipation:
There are many ways to model inelastic collisions that dissipate the relative kinetic energy of colliding particles; e.g. by using Hertz' contact law [55, 56], visco-elastic media [57], or coefficients of normal and tangential restitution [6, 11]. The coefficients of restitution may also depend on the relative speed of the colliding particles [55].

For the ISS-models, studied in this article, the scattering laws (scattering angle, collisional energy loss) are independent of ν, i.e. they are the same for inelastic hard spheres, inelastic Maxwell models and for general ISS-models. Only the collision frequency $\text{A}(\vec{g}, \vec{n})$ of the ISS-models has the same energy dependence as the collision frequency of elastic particles interacting through repulsive power law potentials.

4 Analysis of inelastic scattering models

4.1 Homogeneous cooling laws

We start with the freely evolving case and compute the cooling rate, defined through, $dv_0^2/dt = -\zeta_\nu\,(t)\,v_0^2$. This can be done by applying $\left(\int d\vec{v} v^2\right)$ to the Boltzmann equation (24), changing integration variables $(\vec{v}, \vec{w}) \to (\vec{v}^{\,**}, \vec{w}^{**})$ in the gain term, and using the relations $d\vec{v} d\vec{w} = \alpha d\vec{v}^{\,**} d\vec{w}^{**}$ together with the energy loss ΔE per collision in (15). In general the above cooling equation is a formal identity, and does not provide a closed equation for $v_0(t)$. However, if $F(v, t)$ rapidly relaxes to a scaling form, then the *subsequent* time evolution of $v_0^2 = (2/d)\langle v^2 \rangle$ is described by a closed equation for $v_0(t)$, as we will see, i.e.

$$\frac{dv_0^2}{dt} = \tfrac{2}{d}\int d\vec{v}v^2 I\left(v|F\right) = \tfrac{2}{d}\int_{\vec{n}}\int d\vec{v}d\vec{w}\left|\vec{g}\cdot\vec{n}\right|^{\nu} \Delta EF\left(v,t\right)F\left(w,t\right)$$
$$= -\frac{1}{2d}\left(1-\alpha^2\right)v_0^{\nu+2}\int_{\vec{n}}\int d\vec{c}d\vec{c}_1\left|\left(\vec{c}-\vec{c}_1\right)\cdot\vec{n}\right|^{\nu+2} f\left(c\right)f\left(c_1\right), \quad (37)$$

from which the cooling rate can be identified as,

$$\zeta_\nu\left(t\right) \equiv -\frac{dv_0^2}{dt}/v_0^2 = 2\gamma_0\kappa_{\nu+2}\, v_0^{\nu}\left(t\right). \quad (38)$$

Here the coefficient $\gamma_0 = \frac{1}{4d}\left(1-\alpha^2\right)$ measures the inelasticity. Moreover, the constant κ_ν is defined as,

$$\kappa_\nu = \int_{\vec{n}}\int d\vec{c}d\vec{c}_1\left|\left(\vec{c}-\vec{c}_1\right)\cdot\vec{n}\right|^{\nu} f\left(c\right)f\left(c_1\right) = \beta_\nu\int d\vec{c}d\vec{c}_1\left|\vec{c}-\vec{c}_1\right|^{\nu} f\left(c\right)f\left(c_1\right), \quad (39)$$

and the $\left(d-1\right)-$dimensional angular integral β_ν is evaluated as,

$$\beta_\nu = \int_{\vec{n}}\left|\hat{\vec{a}}\cdot\vec{n}\right|^{\nu} = \frac{\int_0^{\pi/2}d\theta\left(\sin\theta\right)^{d-2}\left(\cos\theta\right)^{\nu}}{\int_0^{\pi/2}d\theta\left(\sin\theta\right)^{d-2}} = \frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{\nu+d}{2}\right)\Gamma\left(\frac{1}{2}\right)}. \quad (40)$$

In one-dimensional systems $\beta_\nu = 1$ for all ν, and in higher dimensions for the ν-values of interest here β_ν is convergent ($\nu > 0$). As a consequence of the scaling ansatz the coefficient κ_ν is a time independent constant, that does depend on the unknown scaling form. For Maxwell models ($\nu = 0$) the cooling rate ζ_0 can be calculated explicitly from (40), and the constant κ_2 is $\kappa_2 = (1/d)\left\langle\left|\vec{c}-\vec{c}_1\right|^2\right\rangle = 1$ on account of (27). The result is,

$$\zeta_0 \equiv -\frac{dv_0^2}{dt}/v_0^2 = 2\gamma_0. \quad (41)$$

The r.m.s. velocity decays then as $v_0\left(t\right) = v_0\left(0\right)\exp\left[-\gamma_0 t\right]$.

The most important time scale in kinetic theory is the mean free time t_{mf}, which equals the inverse of the *mean collision rate* $\omega_\nu\left(t\right)$, defined as the average of the collision frequency $\mathrm{A}\left(\vec{g},\vec{n}\right) = \left|\left(\vec{c}-\vec{c}_1\right)\cdot\vec{n}\right|^{\nu}$ over the velocities of the colliding pair in the scaling state, i.e.

$$\omega_\nu\left(t\right) \equiv -\int d\vec{v}I_{\mathrm{loss}}\left(v|F\right) = \kappa_\nu v_0^{\nu}\left(t\right). \quad (42)$$

Comparison of (38) and (42) shows that both frequencies are related as,

$$\zeta_\nu\left(t\right) = 2\gamma_0\frac{\kappa_{\nu+2}}{\kappa_\nu}\omega_\nu\left(t\right) \equiv 2\gamma_\nu\omega_\nu\left(t\right), \quad (43)$$

where $\gamma_\nu\kappa_\nu = \gamma_0\kappa_{\nu+2}$ are time independent constants. Using (42) and (43) the equation for the r.m.s. velocity becomes, $\dot{v}_0 = -\gamma_\nu\kappa_\nu v_0^{\nu+1}$, yielding a solution, identical to (3) with γ replaced by γ_ν. Consequently the granular temperature at large times decays as $T = v_0^2 \sim t^{-2/\nu}$. For $\nu = 1$ one recovers the well known law of Haff [9], describing the long time decay of the granular temperature in the homogeneous cooling state of inelastic hard spheres. A homogeneous cooling law similar to (3) with $\nu = 6/5$ has been derived in [55], not on the basis of the Boltzmann equation (24) with an energy dependent collision rate $\propto g^{5/6}$, but by using the Boltzmann equation for inelastic hard spheres ($\nu = 1$) with an energy dependent coefficient of restitution $\alpha(g) = 1 - Ag^{1/5}$. For Maxwell models ($\nu \to 0$) the r.m.s. velocity in (3) reduces to $v_0(t) = v_0(0)\exp[-\gamma_0 t]$, in agreement with (41).

The homogeneous cooling law for the general class of ISS-models discussed here, can be cast into a universal form by changing to a new time variable, the *collision counter* or *internal* time τ of a particle, which represents the total number of collisions that a particle has suffered in the (external) time t. It is defined through the mean collision frequency,

$$d\tau = \omega_\nu(t)\, dt, \tag{44}$$

where $\omega_\nu(t) \sim v_0^\nu(t)$. After inserting (3) in this equation the differential equation can be solved to yield,

$$\nu\gamma_\nu\tau = \ln\left[1 + \nu\gamma_\nu\omega_\nu(0)\, t\right], \tag{45}$$

valid for all ISS-models. Combination of (45) and (3) shows that the r.m.s. velocity follows the *universal* homogeneous cooling law,

$$v_0(\tau) = v_0(0)\exp\left[-\gamma_\nu\tau\right], \tag{46}$$

where γ_ν is according to (43) and (39) proportional to the inelasticity γ_0, and depends on ν through the collision integrals κ_ν. We further observe that the relations (45) and (46) also holds for the inelastic BGK-model in Section 2 with γ_ν replaced by $\gamma = \frac{1}{2}\left(1 - \alpha^2\right)$ in (6).

So far we have been dealing with freely cooling systems. Next we address the *balance equation* for the granular temperature in driven cases, where the external input of energy counterbalances the collisional cooling, and may lead to a NESS. We proceed in the same manner as for the free case, and apply $\left(\int d\vec{v}v^2\right)$ to the Boltzmann equation in (30) with the result,

$$\partial_t\left\langle v^2\right\rangle = -\zeta_\nu(t)\left\langle v^2\right\rangle + 2\left\langle \vec{v}\cdot\vec{a}\right\rangle + 2dD, \tag{47}$$

where the first term, $\int d\vec{v}v^2 I(v|F) = -\zeta_\nu(t)\left\langle v^2\right\rangle$, is obtained from (37), (38) and (25). The next two terms are obtained from the driving term in (30), i.e. $\int d\vec{v}v^2\mathcal{F}F(v)$, by performing partial integrations.

The most common ways of driving dissipative fluids [22, 26, 29, 31] is by Gaussian white noise (WN) ($\vec{a} = 0; D \neq 0$), or by a Gaussian thermostat (GT) ($\vec{a} = \gamma\vec{v}; D = 0$), yielding for the balance equations,

$$\frac{dv_0^2}{dt}(t) = \begin{cases} (2\gamma - \zeta_\nu(t))\, v_0^2(t) & \text{(GT)} \\ 4D - \zeta_\nu(t)\, v_0^2(t) & \text{(WN).} \end{cases} \tag{48}$$

Here the collisional loss, $-\zeta_\nu v_0^2$, is counterbalanced by the heat, $2\gamma v_0^2$, generated by the negative friction of the Gaussian thermostat, or by the heat, $4D$, generated by randomly kicking the particles.

As $t \to \infty$ the granular temperature will reach a NESS with $T(\infty) = v_0^2(\infty)$, where the r.m.s. velocity follows from (48) using (38),

$$v_0(\infty) = \begin{cases} (\gamma/\gamma_0\kappa_{\nu+2})^{1/\nu} & \text{(GT)} \\ (2D/\gamma_0\kappa_{\nu+2})^{1/(\nu+2)} & \text{(WN).} \end{cases} \tag{49}$$

With reference to the discussion in Section 2.3 we note that the GT-fixed point solution $v_0(\infty)$ is *attracting* for $\nu > 0$, leading to a stable NESS, and *unstable* for $\nu < 0$ with $v_0(t)$ vanishing at $t \to \infty$ if $v_0(0) < v_0(\infty)$, and diverging if $v_0(0) > v_0(\infty)$. The case $\nu = 0$ is *marginally stable*. For the WN-fixed point similar observations apply with the stability threshold $\nu = 0$ replaced by $\nu = -2$.

4.2 Scaling and non-equilibrium steady states

Free cases:

To investigate the existence of scaling solutions of the Boltzmann equation for freely cooling inelastic systems, we substitute the scaling ansatz (26) into (24), to obtain an integral equation for the scaling form $f(c)$. With the help of (44), (46) and (28) the left hand side of (24) becomes,

$$\begin{aligned} \text{l.h.s.} &= -\dot{v}_0 v_0^{-d-1}\left(\frac{d\tau}{dt}\right)\left\{df(c) + c\frac{d}{dc}f(c)\right\} \\ &= \gamma_\nu\omega_\nu(\tau)\, v_0^{-d}(\tau)\,\nabla_{\vec{c}}\cdot\vec{c}f(c) = \gamma_0\kappa_{\nu+2}\nabla_{\vec{c}}\cdot\vec{c}f(c)\; v_0^{\nu-d}(\tau)\,. \end{aligned} \tag{50}$$

In the last equality we have used the relation $\gamma_\nu\kappa_\nu = \gamma_0\kappa_{\nu+2}$, implied by (43). The resulting integral equation for $f(c)$ becomes,

$$I(c|f) = \gamma_0\kappa_{\nu+2}\nabla_{\vec{c}}\cdot\vec{c}f = \frac{1}{d}\varpi_2\nabla_{\vec{c}}\cdot\vec{c}f. \tag{51}$$

With the help of (37) and (38) the second moment of the collision term, ϖ_2, can be expressed as,

$$\varpi_2 \equiv -\int d\vec{c}c^2 I(c|f) = d\gamma_0\kappa_{\nu+2}. \tag{52}$$

Driven cases:

We consider equation (30) for the NESS distribution in its rescaled form (31) with $v_0 = v_0(\infty)$, driven by a Gaussian thermostat (GT), $\{\vec{a} = \gamma\vec{v}; D = 0\}$ or by white noise (WN), $\{\vec{a} = 0; D \neq 0\}$. The scaling equations for $f(c)$ take the form,

$$I(c|f) = \begin{cases} \dfrac{\gamma}{v_0^\nu} \nabla_{\vec{c}} \cdot \vec{c} f = \dfrac{1}{d} \varpi_2 \nabla_{\vec{c}} \cdot \vec{c} f & \text{(GT)} \\ -\dfrac{D}{v_0^{\nu+2}} \nabla_{\vec{c}}^2 f = -\dfrac{1}{2d} \varpi_2 \nabla_{\vec{c}}^2 f & \text{(WN)}\,. \end{cases} \tag{53}$$

The first equality in (GT) and (WN) suggests that $f(c)$ depends explicitly on γ or D. This is however not the case. The stationarity relation (49) combined with (52) shows in fact that the following expressions,

$$\varpi_2 = d\gamma v_0^{-\nu} = 2dDv_0^{-(\nu+2)} = d\gamma_0 \kappa_{\nu+2}, \tag{54}$$

are *independent* of γ or D. So, we have used (49) to eliminate γ and D, and to put it in the universal form, containing ϖ_2.

4.3 Comments

Equivalence free cooling - Gaussian thermostat:
Comparison of integral equation (51) for free cooling and (53) for the Gaussian thermostat shows that both equations are identical, as first observed by Montanero and Santos [26]. This implies that the scaling form $f(c)$ in free cooling ($\mathcal{F} = 0$) is identical to the NESS distribution $f(c)$ of the same system, driven by a Gaussian thermostat, provided both forms are rescaled to the same constant width $\int d\vec{c} c^2 f(c) = d/2$. This also implies that the scaling form for the free case can be measured by performing Monte Carlo simulations in a steady state [25, 26]. The same idea of systematically rescaling the velocities has also been used in molecular dynamics simulations of a freely cooling system of N inelastic hard spheres [58].

Universality:
The scaling equations (53) for $f(c)$ in systems driven by a Gaussian thermostat and by white noise, have a universal form, because ϖ_2 is independent of the friction constant γ, the noise strength D and the width $v_0(\infty)$, which may depend on the initial distribution. The scaling equations reduce for $\nu = 1$ to those for hard spheres [22], and for $\nu = 0$ to those for Maxwell models [40].

Perturbative approach:
The moment of the collision term $\varpi_2 = d\gamma_0 \kappa_{\nu+2}$ in the scaling equations (53), and the cooling rate $\zeta_\nu = \gamma_0 \kappa_{\nu+2} v_0^\nu$ in (48) contain for all models the quantity κ_ν, as given by (38). It depends on the unknown function $f(c)$, except for $\nu = 0$ where $\kappa_2 = 1$ and $\varpi_2 = d\gamma_0 = \frac{1}{4}\left(1 - \alpha^2\right)$. For the case of inelastic hard spheres, a perturbative method has been developed in [11, 22] for small inelasticities to solve both integral equations in (53) by expanding $f(c)$ in a series of Sonine polynomials $S_p\left(c^2\right)$, i.e.

$$f(c) = \phi(c) \{1 + \sum_{p=2}^{\infty} a_p S_p\left(c^2\right)\}, \tag{55}$$

where $\phi(c) = \pi^{-d/2} \exp\left(-c^2\right)$ is the Maxwellian. Here a_2 is essentially the fourth cumulant which has been calculated explicitly in [22]. For dimensions $d > 1$ it turns out to be proportional to the inelasticity $\left(1-\alpha^2\right)$. However, the one-dimensional case is exceptional because a_2 approaches a finite value for $\alpha \to 1$ [23]. The same method has been successfully applied by Cercignani et al. [31] to an inelastic Maxwell model, and by several authors [25–27, 55] to inelastic hard spheres and related problems. The method can be applied to the inelastic ISS-models as well.

The method focuses on the lower moments of $f(c)$, and the polynomial approximation, cut off after $S_2\left(c^2\right)$, gives a fair representation of $f(c)$ for velocities in the thermal range, $c \leq 2$. The expansion (55) can then be used to calculate κ_ν in (52) in the form $\kappa_\nu = \kappa_\nu^0 + \left(1-\alpha^2\right)\kappa_\nu^1 + \cdots$. For more details we refer to [22]. As an illustration we calculate the lowest approximation κ_ν^0 to κ_ν in (38) using $f(c) \simeq \phi(c)$. The result is,

$$\begin{aligned}\kappa_\nu^0 &= \int_{\vec{n}} \int d\vec{c}d\vec{c}_1 \left|g_\parallel\right|^\nu \phi(c)\,\phi(c_1) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\,|x|^\nu\, e^{-x^2/2} = \frac{2^{\nu/2}}{\sqrt{\pi}} \Gamma\left(\frac{\nu+1}{2}\right). \end{aligned} \tag{56}$$

The lowest approximation to (54) is then $\varpi_2^0 = d\gamma_0\kappa_\nu^0$. The damping rate γ_ν in (43) of the r.m.s. velocity becomes then $\gamma_\nu = (\nu+1)\,\gamma_0$, in agreement with the results for $\nu = 1$ in [22].

4.4 High energy tails

The polynomial expansion (55) describes $f(c)$ only in the thermal range, but contains no meaningful information about velocities c in the asymptotic range ($c \gg 1$). However the high energy tails in the ISS-models with $\nu > 0$ can be determined by a procedure similar to the one used successfully for inelastic hard spheres systems [22], as well as in inelastic Maxwell models driven by white noise [40]. To do so we make the ansatz of stretched Gaussian behavior for the high energy tail, i.e. $f(c) \simeq B \exp\left[-\beta c^b\right]$ with $0 < b < 2$ and $\beta > 0$, and determine b and β by inserting this ansatz in the scaling equation (53) and requiring self-consistency. The border line case $b \to 2$ corresponds to Gaussian tails, and $b \to 0$ suggests power law tails with negative exponents.

An estimate of the rescaled collision term $I(c|f)$ in (24) is made in [22, 40]. This suggests that the loss term is asymptotically dominant over the gain term as long as the exponent b in $\exp\left[-\beta c^b\right]$ is restricted to $b > 0$. This estimate also applies to the ISS-models as long as the exponent $\nu > 0$ and the inelasticity γ_0 is non-vanishing. So the gain term is neglected. Moreover the loss term can be simplified in the asymptotic velocity range. Its dominant contribution comes from collisions with particles having velocities c_1 that are typically in the thermal range ($c_1 = \mathcal{O}(1)$). Consequently the collision rate

$|(\vec{c}-\vec{c}_1)\cdot\vec{n}|^\nu$ for asymptotic dynamics may be replaced by $c^\nu \left|\hat{\vec{c}}\cdot\vec{n}\right|^\nu$, and the total collision term simplifies to,

$$I\left(c|f\right) \sim I_{\mathrm{loss}}\left(c|f\right) \sim -c^\nu \beta_\nu f\left(c\right), \tag{57}$$

with β_ν given by (40). After these preparations we insert (57) and the stretched Gaussian form, $f\left(c\right) \sim \exp\left[-\beta c^b\right]$, into the Boltzmann equation (53), and follow the procedure sketched above. This gives the following universal results (see item 2 below (54)) for the asymptotic high energy tail in $d-$dimensional inelastic $\nu-$models, $f\left(c\right) \sim \exp\left[-\beta c^b\right]$ with

$$\begin{array}{lll} b=0 & \text{inconsistent} & (\text{GT};\ \nu=0) \\ b=\nu & \beta = d\beta_\nu/\nu\varpi_2 & (\text{GT};\ \nu>0) \\ b=\frac{1}{2}\left(\nu+2\right) & \beta=\frac{2}{\nu+2}\sqrt{\frac{2d\beta_\nu}{\varpi_2}} & (\text{WN};\ \nu>0\ ), \end{array} \tag{58}$$

where the scaling functions and high energy tails of GT-driven and freely cooling systems are equivalent.

We conclude this subsection with some comments.

driven systems:

For $0 < b < 2$ both GT- and WN-driving lead to consistent asymptotic solutions of the scaling equations for $\nu > 0$ with overpopulated high energy tails of stretched Gaussian type. We also note, the larger the interactions (larger ν-values) at large impact energies, the smaller the overpopulation of the high energy tails. For $\nu = 2$, corresponding to the inelastic version of the Very Hard Particle model (see end of Section 3.2) we obtain Gaussian behavior ($b = 2$) for both types of thermostats, and there are no longer over-populated tails. In the case of stretched Gaussian tails *all* moments $\int d\vec{c}c^n f\left(c\right) < \infty$. This would not be the case for power law tails.

In the case of white noise driving, the above results with $b = 1+\frac{1}{2}\nu$, include inelastic hard spheres ($\nu = 1$), as well as Maxwell models ($\nu = 0$), and the results coincide with the detailed predictions for $\nu = 1$ [22] and $\nu = 0$ [40]. For GT-driven or freely cooling ISS-models with $\nu > 0$ the tail distributions have a stretching exponent $b = \nu$. The property $\beta b \to 1/\gamma_0$ as $\nu \downarrow 0$ would suggest power law behavior at the stability threshold, $\nu = 0$ (Maxwell models), with an exponent $a = 1/\gamma_0$ (see however next item). The result (58) also includes the exponential decay, $f(c) \sim \exp[-\beta c]$ for inelastic hard spheres, found in [22].

Stability thresholds:

In the previous comments we have speculated on power law tails, in agreement with the exact solution [35] for the one-dimensional freely cooling Maxwell models with $f\left(c\right) \sim c^{-4}$ for $c \gg 1$. As will be shown in section 5, the scaling form for d-dimensional Maxwell models in free cooling does indeed have an algebraic tail $f\left(c\right) \sim 1/c^{d+a}$, *but* with an exponent $a(\alpha) \neq 1/\gamma_0$. However, neither the existence of algebraic tails, nor the possible value of the power law exponents can be obtained legitimately from the results (58). The reason is that the result is not consistent with the *a priori* assumption $I_{\text{gain}} \ll I_{\text{loss}}$ at $c \gg 1$. In fact, gain and loss terms are of the same order of magnitude for $\nu = 0$.

Maxwell models as approximations to hard spheres:
As discussed in item 2 of Section 3.5, inelastic Maxwell models have also been introduced as a sensibly looking mathematical simplification of the Boltzmann equation for inelastic hard spheres. The results of the analysis in the previous section show that the effect of this simplification on the shape of the tail may be very drastic. In the free cooling case the hard sphere tail is exponential, $f(c) \sim \exp[-\beta c]$, whereas in the Maxwell model it is a power law tail, $f(c) \sim 1/c^{d+a}$. A smaller difference exists in the WN-driven case, where the hard sphere tail is $f(c) \sim \exp\left[-\beta c^{3/2}\right]$, and the Maxwell tail is $f(c) \sim \exp[-\beta c]$.

MC-simulations at high and low inelasticity:
The results (58) predict more than the stretching exponent b. In fact the coefficient β for $\nu > 0$ in $f(c) \sim \exp\left[-\beta c^b\right]$ is given in terms of ϖ_2 in (54). For these $\nu-$values we have described a perturbative calculation for ϖ_2, which converges rapidly for $\alpha \to 1$, but gives poor results for $\alpha < 0.6$ [22, 26]. Nevertheless, it is possible to test the predictions of (58) for the GT- and the WN-thermostats, including the coefficient β for *all* values of the restitution coefficient α. This can be done by measuring the mean square velocity $v_0^2(\infty)$ in MC simulations on driven systems in a NESS for a given γ or D. From these data ϖ_2 can be calculated using (54).

5 Inelastic Maxwell models

5.1 Fourier transform method

As discussed in Section 4.3, the behavior of the high energy tails of the scaling form in the ISS-models is only controlled by the loss term in the Boltzmann collision term, and generates the stretched Gaussian tails. However, in the borderline case ($\nu = 0$) the tail behavior is determined by an interplay between gain and loss term, which leads to algebraic tails. This makes the analysis more complicated. In this article we only discuss the borderline case in *free cooling*, formed by the Maxwell models, and we refer to [34] for a more comprehensive discussion.

In this section we want to demonstrate that the Boltzmann equation (24) for inelastic Maxwell models ($\nu = 0$),

$$\partial_t F(v,t) = -F(v,t) + \frac{1}{\alpha} \int_{\vec{n}} \int d\vec{w} F(v^{**},t) F(w^{**},t), \tag{59}$$

has a scaling solution with a power law tail. To do so we first consider the Fourier transform of the distribution function, $\Phi(k,t) = \left\langle \exp\left[-i\vec{k}\cdot\vec{v}\right]\right\rangle$, which is the characteristic function or generating function of the velocity moments. Because $F(v,t)$ is isotropic, $\Phi(k,t)$ is isotropic as well.

As an auxiliary step we first Fourier transform the gain term in (59), i.e.

$$\begin{aligned} &\int d\vec{v} \exp\left[-i\vec{k}\cdot\vec{v}\right] \, I_{\text{gain}}(v|F) = \\ &\int_{\vec{n}} \int d\vec{v} d\vec{w} \exp\left[-i\vec{k}\cdot\vec{v}^{\,*}\right] F(v,t)\, F(w,t) = \int_{\vec{n}} \Phi(k\eta_+,t)\, \Phi(k\eta_-,t)\,. \end{aligned} \quad (60)$$

The transformation needed to obtain the first equality is the same as in (37). Then we use (14) to write the exponent as $\vec{k}\cdot\vec{v}_1^{\,*} = \vec{k}_-\cdot\vec{v}_1 + \vec{k}_+\cdot\vec{v}_2$, where

$$\begin{aligned} \vec{k}_+ &= p\vec{k}\cdot\vec{n}\vec{n} \qquad \left|\vec{k}_+\right| = kp\left|\left(\hat{\vec{k}}\cdot\vec{n}\right)\right| = k\eta_+(\vec{n}) \\ \vec{k}_- &= \vec{k} - \vec{k}_+ \qquad \left|\vec{k}_-\right| = k\sqrt{1 - z\left(\hat{\vec{k}}\cdot\vec{n}\right)^2} = k\eta_-(\vec{n})\,, \end{aligned} \quad (61)$$

with $p = 1-q = \frac{1}{2}(1+\alpha)$ and $z = 1-q^2$. The Fourier transform of (59) then becomes,

$$\partial_t \Phi(k,t) = -\Phi(k,t) + \int_{\vec{n}} \Phi(k\eta_+(\vec{n}),t)\, \Phi(k\eta_-(\vec{n}),t)\,, \quad (62)$$

where $\Phi(0,t) = 1$ because of (25). In one-dimension $\eta_+(\vec{n}) = p$ and $\eta_-(\vec{n}) = q$, and $\int_{\vec{n}}$ can be replaced by unity. Moreover this equation simplifies to [29],

$$\partial_t \Phi(k,t) = \Phi(pk,t)\, \Phi(qk,t) - \Phi(k,t)\,. \quad (63)$$

Because $F(v,t)$ is isotropic, only the even moments are non-vanishing, and the moment expansion of the characteristic function takes the form,

$$\Phi(k,t) = \sum_n{}' \frac{(-ik)^n}{n!} \left\langle \left(\hat{\vec{k}}\cdot\vec{v}\right)^n \right\rangle = \sum_n{}' (-ik)^n\, m_n(t)\,, \quad (64)$$

where the prime indicates that $n = even$, and the moment $m_n(t)$ is defined as,

$$m_n(t) = \beta_n \langle v^n \rangle / n!, \quad (65)$$

where $\beta_n = \int_{\vec{n}} \left(\hat{\vec{k}}\cdot\hat{\vec{v}}\right)^n$ is given in (40). Moreover, the normalizations (25) give $m_0(t) = 1$ and $m_2(t) = \frac{1}{2}\beta_2 \langle v^2 \rangle = \frac{1}{4} v_0^2$.

Maxwell models have the unusual property that the system of moment equations for $m_n(t)$ is closed, and can be solved *sequentially*. The reason is that the rate $\dot{m}_n$ is only a function of lower moments $m_s(t)$ with $s \leq n$.

The moment equations are readily obtained by inserting the expansion (64) in the Fourier transformed Boltzmann equation (62), and equating the coefficients of equal powers of k. The result is,

$$\dot{m}_n + \lambda_n m_n = \sum_{l=2}^{n-2} h\left(l, n-l\right) \; m_l \, m_{n-l} \quad \left(n > 2\right), \tag{66}$$

where all labels $\{n, l, s\}$ take *even* values only. Following [38] or Appendix A of [41] the functions can be calculated for real positive values of ℓ and s with the result,

$$\begin{aligned} h\left(l,s\right) = \int_{\vec{n}} \eta_+^l\left(\vec{n}\right) \eta_-^s\left(\vec{n}\right) = p^l \beta_l \; {}_2F_1\left(-\frac{s}{2}, \frac{l+1}{2}; \frac{l+d}{2} \mid z\right) \\ \lambda_s =& 1 - h\left(s,0\right) - h\left(0,s\right) = \int_{\vec{n}} \left[1 - \eta_+^s\left(\vec{n}\right) - \eta_-^s\left(\vec{n}\right)\right] \\ =& 1 - p^s\beta_s - \; {}_2F_1\left(-\frac{s}{2}, \frac{1}{2}; \frac{d}{2} \mid 1-q^2\right). \end{aligned} \tag{67}$$

Here ${}_2F_1(\alpha, \beta; \gamma \mid z)$ is a hypergeometric function, and $\lambda_2 = 2pq/d = 2\gamma_0$ is easily obtained from the above results.

Next we consider the Fourier transform of the scaling relation (26), yielding $\Phi\left(k,t\right) = \phi\left(kv_0\left(t\right)\right)$ with $v_0\left(t\right) = v_0\left(0\right)\exp\left(-\gamma_0 t\right)$ according to (41). Inserting Φ in (62) gives the integral equation for the scaling form $\phi\left(k\right)$, which reads

$$-\gamma_0 k \frac{d}{dk}\phi\left(k\right) + \phi\left(k\right) = \int_{\vec{n}} \phi\left(k\eta_+\right)\phi\left(k\eta_-\right). \tag{68}$$

For $d = 1$ it reduces to,

$$-pqk\frac{d}{dk}\phi\left(k\right) + \phi\left(k\right) = \phi\left(pk\right)\phi\left(qk\right). \tag{69}$$

The scaling form $\phi\left(k\right)$ is the generating function for the moments of $f\left(c\right)$, i.e.

$$\begin{aligned} \phi\left(k\right) =& \sum_n{}' \frac{\left(-ik\right)^n}{n!}\beta_n\left\langle c^n\right\rangle \equiv \sum_n{}' \left(-ik\right)^n \mu_n \\ \simeq& 1 - \tfrac{1}{4}k^2 + k^4\mu_4 - k^6\mu_6 + \cdots, \end{aligned} \tag{70}$$

where n =even, $\mu_0 = 1$ and $\mu_2 = \frac{1}{2}\beta_2\left\langle c^2\right\rangle = 1/4$ on account of the normalizations (27) and $\beta_2 = 1/d$ (see (40)). By inserting (70) into (68) one obtains the recursion relation,

$$\mu_n = \frac{1}{\lambda_n - \frac{1}{2}n\lambda_2}\sum_{l=2}^{n-2} h\left(l, n-l\right)\mu_l\mu_{n-l} \qquad \left(n > 2\right) \tag{71}$$

with initialization $\mu_2 = 1/4$ and all labels $n, l = even$. How these moments behave as a function of α has been calculated in [41] by numerically solving the recursion relation.

5.2 Small-k singularity of the characteristic function

The asymptotic analysis in Section 4.4 of the high energy tail $f(c) \sim \exp\left[-\beta c^b\right]$ for Maxwell models ($\nu = 0$) yields $b = 0$, which suggests a power law tail $f(c) \sim 1/c^{d+a}$ as the leading large$-c$ behavior in $d-$dimensional systems. If this is indeed the case, then the moments μ_n of the scaling form $f(c)$ are convergent if $n < a$ and are divergent if $n > a$. As we are interested in physical solutions which can be *normalized*, and have a *finite energy*, a possible value of the power law exponent must obey $a > 2$.

The characteristic function is in fact a very suitable tool for investigating this problem. Suppose the moment μ_n with $n > a$ diverges, then the n-th derivative of the corresponding generating function also diverges at $k = 0$, i.e. $\phi(k)$ has a singularity at $k = 0$. Then a simple rescaling argument of the inverse Fourier transform shows that $\phi(k)$ has a dominant small-k singularity of the form $\phi(k) \sim |k|^a$, where $a \neq$ *even*. On the other hand, whenever the exponent b in $f(c) \sim \exp\left[-\beta c^b\right]$ is positive, – as is the case for the inelastic Maxwell model *driven by white noise*, where $b = 1$ and $\beta = \sqrt{2/pq}$ – , then all moments are finite, and the characteristic function $\phi(k)$ is regular at the origin, i.e. can be expanded in powers of k^2.

We first illustrate our analysis for the one-dimensional case. As the requirement of finite total energy imposes the lower bound $a > 2$ on the exponent, we make the ansatz, consistent with (70) that the dominant small-k singularity has the form,

$$\phi(k) = 1 - \tfrac{1}{4}k^2 + A\,|k|^a\,, \tag{72}$$

insert this in (69), and equate the coefficients of equal powers of k. This yields,

$$\tfrac{1}{2}a\lambda_2 = \lambda_a \quad \text{or} \quad apq = 1 - p^a - q^a. \tag{73}$$

The equation has two roots, $a = 2, 3$, of which $a = 3$ is the one larger than 2. Here A is left undetermined. Consequently the one-dimensional scaling solution has a power law tail, $f(c) \sim 1/c^4$, in agreement with the exact solution in [35].

For general dimension we proceed in the same way as in the one-dimensional case, insert the ansatz (72) into (68), and equate the coefficients of equal powers of k. This yields for the coefficient of k^2 the identity $2\gamma_0 = \lambda_2$, and for the coefficient of k^a the transcendental equation,

$$\tfrac{1}{2}a\lambda_2 = \lambda_a = \textstyle\int_{\vec{n}}\left[1 - \eta_+^a - \eta_-^a\right]. \tag{74}$$

The equation above obviously has the solution $a = 2$. We are however interested in the solution with $a > 2$. In the elastic limit ($\alpha \to 1$) the solution is simple. There $\gamma_0 \to 0$ and a diverges. The contributions of $\eta_\pm^a$ on the right hand side vanish because $\eta_\pm < 1$, and the result is,

$$a \simeq \frac{1}{\gamma_0} = \frac{4d}{1-\alpha^2}. \tag{75}$$

For general values of α one can conveniently use an integral representation of ${}_2F_1$ to evaluate λ_a and solve the transcendental equation (74) numerically. We illustrate the solution method of (74) with the graphical construction in Figure 2, where we look for intersections of the line $y = \frac{1}{2}\lambda_2 = \gamma_0 s$ with the curve $y = \lambda_s$ for different values of α.

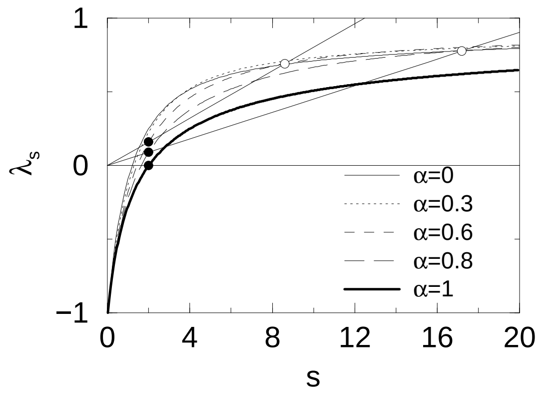

Fig. 2. Graphical solution of Eq.(74) for different values of the parameter α. The eigenvalue λ_s is a concave function of s, plotted for different values of the restitution coefficient α for the 2-D inelastic Maxwell model. The line $y = s\gamma_0$ is plotted for $\alpha = 0.6, 0.8$ and $\alpha = 1$(top to bottom). The intersections with λ_s determine the points s_0 (filled circles) and s_1 (open circles). Here $s_1 = a$ determines the exponent of the power law tail. For the elastic case ($\alpha = 1, \gamma_0 = 0$, energy conservation) there is only one intersection point.

The relevant properties of λ_s are: (i) $\lim_{s\to 0} \lambda_s = -1$; (ii) λ_s is a concave function, monotonically increasing with s, and *(iii)* all eigenvalues for positive *integers* n are positive (see Figure 2). As can be seen from the graphical construction, the transcendental equation (74) has two solutions, the trivial one ($s_0 = 2$) and the solution $s_1 = a$ with $a > 2$. The numerical solutions for $d = 2, 3$ are shown in Figure 3 as a function of α, and the α-dependence of the root $a(\alpha)$ can be understood from the graphical construction. In the elastic limit as $\alpha \uparrow 1$ the eigenvalue $\lambda_2(\alpha) \to 0$ because of energy conservation. In that limit the transcendental equation (74) no longer has a solution with $a > 2$, and $a(\alpha) \to \infty$ according to (75), as it should be. This is consistent with a Maxwellian tail distribution in the elastic case. Krapivsky and Ben-Naim have in fact solved the transcendental equation asymptotically for large d, which gives qualitatively the same results as shown in Figure 3 for two and three dimensions.

These results establish the existence of scaling solutions $f(c) \sim 1/c^{d+a}$ with algebraic tails, where the exponent a is the solution of the transcendental

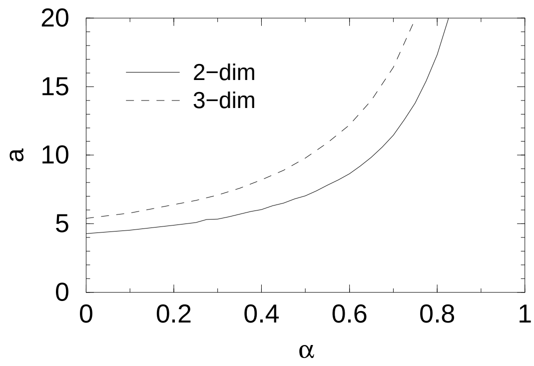

Fig. 3. Exponent $a(\alpha)$, which is the non-trivial root of (74), shown as a function of the coefficient of restitution α, and denoted in the plot by α. It determines the high energy tail $1/c^{a+d}$ of the scaling solution $f(c)$ of the 2-D and 3-D Maxwell models.

equation (74) with $a > 2$. The exponent $a(\alpha)$ behaves as a function of α qualitatively the same as in the simple inelastic BGK model of Section 2.

Using a somewhat different analysis Krapivsky and Ben-Naim [37] independently obtained the same results for the algebraic tails in freely cooling Maxwell models.

5.3 Beyond asymptotic analysis

Free cooling:
As mentioned already in Section 3.3, Baldassarri et al. [35] have obtained an exact solution $f(c)$ of the scaling equation for the one-dimensional Maxwell model in free cooling, and they demonstrated the importance of this scaling solution by means of Monte Carlo simulations. In doing so they found that $F(v,t)$ for different classes of initial distributions can be collapsed for *long* times t on this exact scaling solution $f(c)$, when $v_0(t)\,F(v,t)$ is plotted versus $c = v/v_0(t)$.

We briefly illustrate here how this solution is obtained from the one-dimensional scaling equation (69). One first verifies that the following function,

$$\phi(k) = (1 + \vartheta\,|k|)\exp(-\vartheta\,|k|)\,, \tag{76}$$

with arbitrary positive ϑ is a solution, that can be Fourier inverted. It gives the scaling form,

$$f(c) = \frac{2}{\pi\vartheta}\frac{1}{(1 + c^2/\vartheta^2)^2}\,. \tag{77}$$

To determine the scaling form that satisfies the normalizations (27), we expand (76) in powers of $|k|$ to obtain,

$$\phi(k) = 1 - \tfrac{1}{2}\vartheta^2 k^2 + \tfrac{1}{3}\vartheta^3 |k|^3 + \cdots . \tag{78}$$

Comparison of this result with (70) shows that $\vartheta = 1/\sqrt{2}$. Moreover, it confirms the ansatz (72) used to find solutions with small-k singularities. Comparison also shows the value of the coefficient $A = 1/\left[6\sqrt{2}\right]$ in (72), and the high energy tail is $f(c) \sim \mathcal{A}/c$ with $\mathcal{A} = 1/\left[\pi\sqrt{2}\right]$. The coefficients A and $\mathcal{A}$ can not be determined within the asymptotic method.

White noise driving:
For the one-dimensional Maxwell model driven by white noise the steady state solution has also been found exactly with the help of the Fourier transform method [29]. The characteristic function satisfies in that case a nonlinear finite difference equation, which can be solved by iteration. However, the analytic structure is rather complex, and makes it difficult to extract analytic information from that solution.

The observation that $f(c) \sim \exp[-\beta|c|]$ has an exponential high energy tail, was probably first made in numerical work of van der Hart and Nienhuis [43]. A more detailed analytic prediction about the asymptotic tail was given in [40], where it was shown that $\beta = \sqrt{8/(1-\alpha^2)}$ for all Maxwell models, independent of the dimensionality ($d = 1, 2, \cdots$), at least with the normalization, $\langle c^2 \rangle = \frac{1}{2}d$, used in this article. This result follows directly from (58) and the value $\varpi_2 = d\gamma_0 = \frac{1}{4}\left(1-\alpha^2\right)$, given in item 3 below (54). Furthermore, additional numerical and analytical work was also published by Marconi and Puglisi [59], and by Antal et al. [44]. Only recently more detailed analytic results have been extracted from the rather complex structure of the exact solution [45, 53].

6 Conclusions and perspectives

We have studied asymptotic properties of scaling or similarity solutions, $F(v,t) = (v_0(t))^{-d} f(v/v_0(t))$, of the nonlinear Boltzmann equation in spatially homogeneous systems composed of particles with *inelastic* interactions for large times and large velocities. The large $t-$ and $v-$scales are relevant because on such scales the universal features of the solutions survive, while details of the initial distributions, of interaction strength, and degree of inelasticity, are mostly lost. The behavior of these scaling states, which describe nonequilibrium steady states (NESS), is less universal than the state of thermal equilibrium, because the form of the NESS distribution $f(c)$ depends on the way of driving the dissipative systems. Scaling solutions are very well suited to expose the universal features of the velocity distribution functions, because the velocities, $c = v/v_0(t)$, are measured in units of the r.m.s. velocity or instantaneous width $v_0(t)$ of the distribution.

The real importance of the scaling solutions is that the actual solutions $F(v,t)$ for large classes of initial distributions $F(v,0)$ (essentially initial data without over-populated tails -see [41]) rapidly approach these scaling solutions in the sense that after a short transient time the data $v_0^d(t) F(v,t)$ can be collapsed on a single scaling form $f(c)$, as first observed by Baldassarri et al. in MC simulations of the nonlinear Boltzmann equation for a one-dimensional inelastic Maxwell model. This rapid approach to a universal scaling function applies both to systems driven by a Gaussian or by a white noise thermostat, as well as for freely cooling systems, which show scaling behavior, identical to systems driven by a Gaussian thermostat.

Originally this scenario had the status of a conjecture for systems of inelastic hard spheres ($\nu = 1$) and inelastic Maxwell models ($\nu = 0$), as formulated in [41] where also some analytical evidence for the approach to a scaling form has been presented. For Maxwell models the conjecture has been rigorously proven in the mean time by Bobylev et al. [42]. The analysis in Section 4.1 also suggests what the basic criterion is for the approach of ISS-models to the scaling form. If the energy balance equation (48) has a stable/attractive fixed point solution, $v_0(\infty)$, then the distribution function $F(v,t)$ approaches – in the sense detailed in the paragraphs above – a stable NESS described by the scaling solution with a high energy tail that is generically of stretched Gaussian type.

The analysis in Section 4.1 of the energy balance equation also suggests that cases of marginal stability – which are the exceptional border line cases – are candidates for power law tails, where the freely evolving inelastic Maxwell model is a well known example. At what $\nu-$value an ISS-model is *marginal* depends on the type of energy input or thermostat. A further case of power law tails is the ISS-model with $\nu = -2$, which is the marginal case for white noise forcing. This, and the relation between marginal stability and power law tails, as well as additional cases of power law tails are further explored and verified by MC simulations in [34]. The general conclusion is here that among ISS-models stretched Gaussian tails represent the generic form of high energy tails, and that power law tails are exceptions.

In this article we have focused on the properties of the scaling solution, and in particular on its high energy tail. Here we have introduced the Boltzmann equation for new classes of inelastic interactions, named Inelastic Soft Spheres or ISS-models, corresponding to pseudo-repulsive power law potentials, with collision probabilities proportional to $\mathrm{A} \sim |\vec{g} \cdot \vec{n}|^\nu$, covering hard scatterers like inelastic hard spheres ($\nu = 1$) and soft scatterers like pseudo-Maxwell molecules ($\nu = 0$). The energy loss in an inelastic interaction is proportional to the inelasticity, $\gamma_0 \sim \left(1-\alpha^2\right)$, where α is the coefficient of restitution. We have studied two typical cases: a freely evolving system in homogeneous cooling state, without energy supply, and systems with energy supply, driven by Gaussian thermostats with negative friction or driven by Gaussian white noise.

The homogeneous cooling laws in these systems are described by the granular temperature $T \propto v_0^2$, with a long time decay as $t^{-2/\nu}$ or $\exp[-2\gamma_\nu \tau]$, where t is the external (laboratory) time, and τ is the internal time or collision counter, and γ_ν is proportional to the inelasticity γ_0.

An interesting feature of all ISS-models is that their scaling forms generically have a stretched Gaussian high energy tail, $f(c) \sim \exp[-\beta c^b]$. In *freely cooling* ISS-models with $\nu > 0$ the stretching exponent $b = \nu$ satisfies $0 < b = \nu \leq 2$. Among the ISS-models at the stability threshold only the freely cooling Maxwell model has been analyzed in the present paper, and it does yield power law tails, $f(c) \sim 1/c^{d+a}$, and the exponent a has been calculated from a transcendental equation. In the ISS-models with $\nu > 0$, driven by white noise, there exist again stretched Gaussian tails with $b = \frac{1}{2}(\nu + 2)$. The general conclusion is here: the larger the $\nu-$values, i.e. the larger the scattering cross-sections or collision frequencies at high energy, the closer the stretching exponent is to $b = 2$, i.e. the closer the tails are to Gaussian tails. All known results for higher dimensional inelastic hard spheres ($\nu = 1$) and inelastic Maxwell models ($\nu = 0$) are recovered in the present analysis.

In ISS-models with positive ν the tails are always stretched Gaussians, and they are determined only by the loss term in the Boltzmann equation. In the freely cooling Maxwell model, and more generally at a stability threshold – which are the cases leading to power law tails [34] – the loss and gain term in the nonlinear Boltzmann equation are of comparable size, and partially balancing each other.

We have also analyzed in Section 2 an inelastic BGK-model, introduced by Brey et al [49], and generalized it to an energy dependent collision frequency, and we added energy source terms to this kinetic equation as well. In all cases the scaling solution can be calculated exactly. The freely cooling model shows an algebraic tail, $f(c) \sim 1/c^{d+a}$ with an exponent $a \sim 1/\gamma \sim 2/\left(1-\alpha^2\right)$. For white noise driving one finds asymptotically $f(c) \sim \exp[-\beta c]$ with $\beta \sim \left(1-\alpha^2\right)^{-1/2}$. In this BGK-model the existence of algebraic tails, as well as the value of the tail exponents are independent of the energy dependence of the collision frequency in this model.

It is interesting to observe that the overpopulated high energy tails in the inelastic BGK-models, both for free cooling as well as for white noise driving, are essentially the same as for the more complicated Maxwell models. However, the present extension to an energy-dependent collision frequency does not capture the generic features of the less-singular stretched Gaussian tails for the ISS-models with $\nu > 0$, as predicted by the nonlinear Boltzmann equation for these models.

Acknowledgements

The authors want to thank J.W. Dufty, A. Santos, E. Trizac, A. Barrat and K. Shundyak for helpful discussions and correspondence. This work is supported by DGES (Spain), Grant No BFM-2001-0291.

References

1. H. Waldmann, in: *Handbuch der Physik*, Vol 12, S. Flügge (Ed.) (Springer-Verlag, 1958).
2. S. Chapman and T.G. Cowling, *Mathematical Theory of Non-Uniform Gases*, (Cambridge Univ. Press, 1970).
3. P. Résibois and M. de Leener, *Classical Kinetic Theory of Fluids* (Wiley, New York, 1977).
4. C. Cercignani, *The Boltzmann Equation and its Applications* (Springer Verlag, New York, 1988).
5. G.E. Uhlenbeck and G.W. Ford, *Lectures in Statistical Mechanics* (Americam Math. Society, Providence, Rhode Island, 1963).
6. C.S. Campbell, Annu. Rev. Fluid Mech. **22**, 57 (1990).
7. J.T. Jenkins and M.W. Richman, J. Fluid Mech. **192**, 313 (1988).
8. J.T. Jenkins and S.B. Savage, J. Fluid Mech. **130**, 187 (1985).
9. P.K. Haff, J. Fluid. Mech. **134**, 40 (1983).
10. N. Sela and I. Goldhirsch, Phys. Fluids **7**, 507 (1995).
11. A. Goldshtein and M. Shapiro, J. Fluid Mech. **282**, 75 (1995).
12. I. Goldhirsch and G. Zanetti, Phys. Rev. Lett. **70**, 1619 (1993). I. Goldhirsch, M.L. Tan and G. Zanetti, J. Scientific Computing **8**, 1 (1993).
13. S.R. Nagel, Rev. Mod. Phys. **64**, 321 (1992).
14. S. Luding, R. Cafiero and H.J. Herrmann, this volume, page 287.
15. A. Baldassarri, U. Marini Bettolo Marconi, and A. Puglisi, this volume, page 93.
16. J.J. Brey, J.W. Dufty and M.J. Ruiz-Montero, this volume, page 225.
17. I. Goldhirsch and T. P. C. van Noije, Phys. Rev. E **61**, 3241 (2000).
18. M.O. Cáceres, Phys. Rev. E **67**, 016102 (2003).
19. D.R.M. Williams and F.C. MacKintosh, Phys. Rev. E **54**, R9 (1996).
20. J.J. Brey, J.W. Dufty and A. Santos, J. Stat. Phys. **87**, 1051 (1997).
21. S.E. Esipov and T. Pöschel, J. Stat. Phys. **86**, 1385 (1997).
22. T.P.C. van Noije and M.H. Ernst, Granular Matter **1**, 57 (1998), and cond-mat/9803042.
23. A. Barrat, T. Biben, Z. Rácz, E. Trizac, and F. van Wijland, J. Phys. A: Math. Gen. **35**, 463 (2002) and cond-mat/0110345.
24. Th. Biben, Ph. A. Martin and J. Piasecki, Physica A **310**, 308 (2002).
25. J.J. Brey, M.J. Ruiz-Montero and D. Cubero, Phys. Rev. E **54** 3664 (1996). J.J. Brey, D. Cubero and M.J. Ruiz-Montero, Phys. Rev. E **59**, 1256 (1999).
26. J.M. Montanero and A. Santos, Granular Matter **2**, 53 (2000) and cond-mat/0002323.
27. M. Huthmann, J.A.G. Orza and R. Brito, Granular Matter **2**, 189 (2000), and cond-mat/0004079.
28. I. Pagonabarraga, E. Trizac, T.P.C. van Noije and M.H. Ernst, Phys. Rev. E **65**, 011303 (2002) and cond-mat/0107570.
29. E. Ben-Naim and P. Krapivsky, Phys. Rev. E **61**, R5 (2000).
30. A.V. Bobylev, J.A. Carrillo, and I.M. Gamba, J. Stat. Phys. **98**, 743 (2000); see also erratum: J. Stat. Phys. **103**, 1137 (2001).
31. J.A. Carrillo, C. Cercignani, and I.M. Gamba, Phys. Rev. E **62**, 7700 (2000); C. Cercignani, J. Stat. Phys. **102**, 1407 (2001).
32. G.P. Collins, Sci. Am. **284**(1), 17 (2001).

33. F. Rouyer and N. Menon, Phys. Rev. Lett. **85**, 3676 (2000); D.L. Blair and A. Kudrolli, Phys. Rev. E **64**, 050301(R) (2001); C. Bizon, M.D. Shattuck, J.B. Swift, and H.L. Swinney, Phys. Rev. E **6**, 4340 (1999); J.S. Olafsen and J.S. Urbach, Phys. Rev. Lett. **81**, 4369 (1998); Phys. Rev. E **60**, R2468 (1999); W. Losert, D.G.W. Cooper, J. Delour, A. Kudrolli and J.P. Gollub, Chaos **9**, 682 (1999); I.S. Aranson and J.S. Olafsen, Phys. Rev. E **66**, 061302 (2002).
34. M.H. Ernst, E. Trizac and A. Barrat, to appear.
35. A. Baldassarri, U. Marini Bettolo Marconi, and A. Puglisi, Europhys. Lett. **58**, 14 (2002), and this volume, page 93.
36. A. Baldassarri, U. Marini Bettolo Marconi, and A. Puglisi, Phys. Rev. E **65**, 051301 (2002).
37. P.L. Krapivsky and E. Ben-Naim, J. Phys. A: Math Gen. **35**, L147 (2002), and this volume, page 63.
38. E. Ben-Naim and P. Krapivsky, Phys. Rev. E **66**, 11309 (2002).
39. M.H. Ernst and R. Brito, cond-mat/0111093, 6 Nov. 2001, and Europhys. Lett. **58**, 182 (2002).
40. M.H. Ernst and R. Brito, Phys. Rev. E **65**, 040301(R) (2002).
41. M.H. Ernst and R. Brito, J. Stat. Phys. **109**, 407 (2002).
42. A.V. Bobylev and C. Cercignani, J. Stat. Phys. **106**, 547 (2002); A.V. Bobylev, C. Cercignani and G. Toscani, J. Stat. Phys. **111**, 403 (2003).
43. B. Nienhuis and R. van der Hart, unpublished; R. van der Hart, *Road Models and Granular Gases*, Master's thesis, Universiteit van Amsterdam, May 24, 2002.
44. T. Antal, M. Droz and A. Lipowski, Phys. Rev. E **66**, 062301(2002).
45. A. Santos and M.H. Ernst, cond-mat/0302285.
46. A.V. Bobylev, Sov. Phys. Dokl. **20**, 820 (1975); A.V. Bobylev, Sov. Phys. Dokl. **21**, 632 (1976).
47. T. Krook and T.T. Wu, Phys. Rev. Lett. **36**, 1107 (1976).
48. P.L. Bhatnagar, E.P. Gross and M. Krook, Phys. Rev. A **94**, 511 (1954).
49. J.J. Brey, F. Moreno, J.W. Dufty , Phys. Rev. A **54**, 445 (1996).
50. S. Ulam, Adv. Apll. Math. **1**, 7 (1980).
51. M.H. Ernst, Phys. Rep. **78**, 1 (1981).
52. T.P.C. van Noije and M.H. Ernst, in: *Granular Gases*, T. Pöschel and S. Luding (Eds), Lecture Notes in Physics (Springer-Verlag, Berlin, 2001), p1-30. M.H. Ernst, in: *Dynamics: models and kinetic methods for non-equilibrium many body systems*, J. Karkheck, (Ed.), Nato Science Series E-Vol. 371, (Kluwer Academic Publs, Dordrecht, 1998) p.239-266.
53. E. Ben-Naim and P.L. Krapivski, this volume, page 63.
54. J. R. Dorfman, *An Introduction to Chaos in Nonequilibrium Statistical Mechanics*, Cambridge Lecture Notes in Physics, Vol 14, section 13.4 (Cambridge University Press, 1999).
55. N.V. Brilliantov and T. Pöschel, in: *Granular Gases*, T. Pöschel and S. Luding (Eds), Lecture Notes in Physics (Springer-Verlag, Berlin, 2001), p.100-124; R. Ramírez, T. Pöschel, N.V. Brilliantov and T. Schwager, Phys. Rev. E **60**, 4465 (1999).
56. S. Chen, Y. Deng, X. Nie and Y. Tu, Phys. Lett. A **269**, 218 (2000).
57. H. Hwang and K. Hutter, Continuum Mech. and Thermodynamics **7**, 357 (1995).
58. R. Soto, M. Mareschal and M. Malek Mansour, Phys. Rev. E **62**, 3836 (2000).
59. U.M. Bettolo Marconi, and A. Puglisi, Phys. Rev. E 66, 011301 (2002).