Paradoxical games, ratchets, and related phenomena

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1   Introduction

Fluctuations and noise have been a major topic in statistical mechanics since Einstein's works on Brownian motion. The theory of thermal fluctuations helped to understand noise in electrical circuits, activation processes in chemistry, the statistical nature of the Second Law and the Maxwell demon, and the origin of critical phenomena and spontaneous symmetry breaking, to cite only a few examples. In most of these cases, the role played by thermal fluctuations or thermal noise is either to trigger some process or to act as a disturbance. However, in the past two decades, the study of fluctuations has led to models and phenomena where the effect of noise is more complex and sometimes unexpected and even counterintuitive.

Noise can enhance the response of a nonlinear system to an external signal, a phenomenon known as stochastic resonance [1]. It can create spatial patterns and ordered states in spatially extended systems [2, 3, 4]. Brownian ratchets show that noise can be rectified and used to induce a systematic motion in a Brownian particle [5, 6, 7]. In these new phenomena, noise has a very different role from that considered in the past: it contributes to the creation of order. This could be relevant in several fields, and specially in biology, since most biological systems manage to keep themselves in ordered states even while surrounded by noise, both thermal noise at the level of the cell and environmental fluctuations at the macroscopic level.

The so-called *Parrondo's paradox* is a rather simple illustration of an unexpected feature of noise, and shows the basic mechanisms underlying a Brownian ratchet. In fact, the paradox came up as a translation to gambling games of the flashing ratchet discovered by Ajdari and Prost [5].

The paradox consists of two games with the following peculiarity: in both of them the player has a systematic tendency to lose but, if they are alternated, the resulting game becomes winning [8, 9, 10]. The games reveal that the outcome

of the alternation of two stochastic dynamics can significantly differ from each separate one.

To grasp an intuitive understanding of this paradox, one can distinguish three mechanisms which act simultaneously in the games but inspire different extensions and applications. Firstly, the alternation of the two dynamics can stabilize the transient states of each separate dynamics, something that can happen even in deterministic dynamics. Secondly, as we have already mentioned, the paradox is an example of a noise rectifier, i.e., of a system that eliminates negative fluctuations and "promotes" the positive ones. Thirdly, one can explain the paradox as a reorganization of the trends present in one of the games.

The paper is organized as follows. In section 2 we briefly review the flashing ratchet and explain how it can rectify fluctuations. Section 3 is devoted to the original Parrondo's paradox. There we introduce the paradoxical games as a discretization of the flashing ratchet, discuss the reorganization of trends mentioned above, and present an extension of the original paradox inspired by this idea. In section 4 we introduce several versions of the games involving a large number of players. Some interesting effects can be observed in these collective games: redistribution of capital brings wealth [11], and collective decisions taken by voting or by optimizing the returns in the next turn can lead to worse performance than purely random choices [12, 13]. In section 5 we turn to a very different field: pattern formation in spatially extended systems. The general idea that switching between two dynamics stabilizes transient states is used in this section to design a new mechanism for pattern formation [14, 15, 16]. Finally, in section 6 we briefly review the literature on the paradox and present our main conclusions.

2 Ratchets

Here we revisit the flashing ratchet [5, 6], one of the simplest Brownian ratchets and the most closely related to the paradoxical games. We refer to the exhaustive review by Reimann on Brownian ratchets [7] or the special issue in Applied Physics A, edited by Linke [17], for further information on the subject.

Consider an ensemble of independent one-dimensional Brownian particles in the asymmetric sawtooth potential depicted in Fig. 1. It is not difficult to show that, if the potential is switched on and off periodically, the particles exhibit an average motion to the right. Let us assume that the temperature T is low enough to ensure that kT is much smaller than the maxima of the potential, and that we start with the potential switched on and with all the particles around one of its minima, as shown in the upper plot of Fig. 1. When the potential is switched off, the particles diffuse freely, and the density of particles spreads as depicted in the central plot of
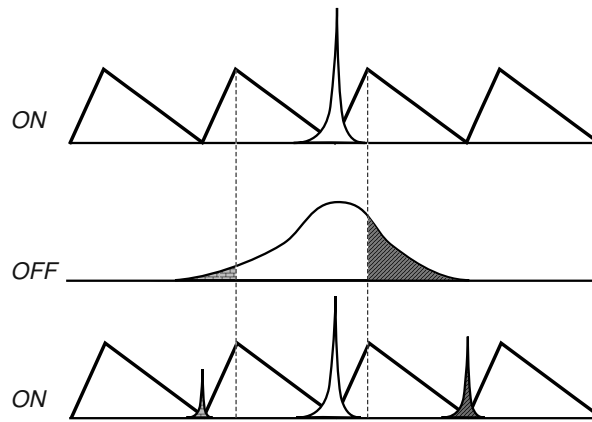
Figure 1: The flashing ratchet at work. The figure represent three snapshots of the potential and the density of particles.

the figure. If the potential is then switched on again, each particle will move back to the initial minimum or to one of the nearest neighboring minima, depending on its position. Particles within the dark region will move to the right hand minimum, those within the small grey region will move to the left hand minimum, and those within the white region will move back to their initial positions. As is apparent from the figure and due to the asymmetry of the potential, more particles fall into the right hand minimum, and there is thus a net motion of particles to the right. For this to occur, the switching can be either random or periodic, but the average period must be of the order of the time to surmount the nearest barrier by free diffusion (see [5, 6] for details).

This motion can be seen as a rectification of the thermal noise associated with free diffusion. The diffusion is symmetric: some particles move to the right and some to the left, but their average position does not change. However, when the potential is switched on again, most of the particles that moved to the left are driven back to the starting position, whereas many particles that moved to the right are pushed to the right hand minimum. The asymmetric potential acts as a rectifier: it "kills" most of the negative fluctuations and "promotes" most of the positive ones.

The effect remains if we add a small force toward the left, i.e., in a direction opposite to the induced motion. In this case, the ratchet still induces a motion against the force. Consequently, particles perform work, and the system can be considered a Brownian motor. It can be proved that this type of motor is compatible with the Second Law of thermodynamics. In fact, the efficiency of such a motor

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is far below the limits imposed by the Second Law [18, 19]. However, the ratchet with a force exhibits a curious property: when the potential is permanently on or off, the Brownian particles move in the same direction as the force, whereas they move in the opposite direction when the potential is switched on and off.

We have seen that the ratchet effect can account for this surprising behavior. An alternative interpretation is the following. The stationary state of the Brownian particles when the potential is permanently on or off possesses a negative velocity. On the other hand, when the potential is switched on and off periodically, the system cannot reach these stationary states, but oscillates between transient states of each dynamics (in fact, to have a net motion to the right, only the transient of the free diffusion is necessary). Therefore, the switching of dynamics stabilizes a behavior which is only transient in each separate dynamics.

3 Games

The flashing ratchet can be discretized in time and space, keeping most of its interesting features. The discretized version adopts the form of a pair of simple gambling games, which are the basis of the Parrondo's paradox.

3.1 The original paradox

We consider two games, A and B, in which a player can make a bet of 1 euro. $X(t)$ denotes the capital of the player, where $t = 0, 1, 2 \ldots$ stands for the number of turns played. Game A consists of tossing a slightly biased coin so that the player has a probability p_A of winning which is less than a half. That is, $p_A = 1/2 - \epsilon$, where the bias ϵ is a small positive number.

The second game, B, is played with two biased coins, a "bad coin" and a "good coin". The player must toss the bad coin if her capital $X(t)$ is a multiple of 3, the probability of winning being $p_{\mathrm{bad}} = 1/10 - \epsilon$. Otherwise, the good coin is tossed and the probability of winning is $p_{\mathrm{good}} = 3/4 - \epsilon$. The rules of games A and B are represented in Fig. 2, in which the darkness represent the "badness" of each coin.

For these choices of p_A, p_{good} and p_{bad}, both games are fair if $\epsilon = 0$, in the sense that $\langle X(t) \rangle$ is constant. This is evident for game A, since the probabilities to win and lose are equal. The analysis of game B is more involved, but we will soon prove that the effect of the good and the bad coins cancel each other for $\epsilon = 0$.

On the other hand, both games have a tendency to lose if $\epsilon > 0$, i.e., $\langle X(t) \rangle$ decreases with the number of turns t. Surprisingly enough, if the player randomly chooses the game to play in each turn, or plays them following some predefined
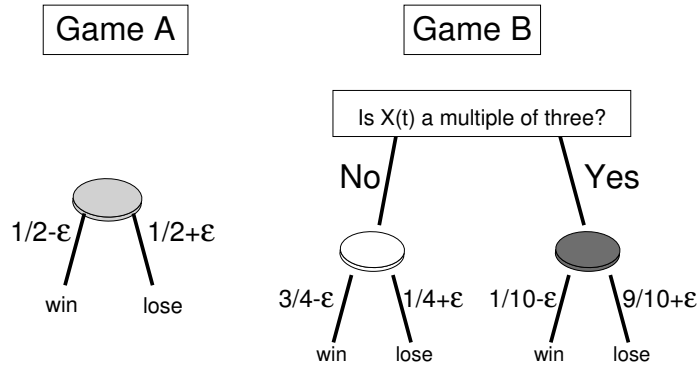
Figure 2: Rules of the paradoxical games.

periodic sequence such as ABBABB..., then her average capital $\langle X(t) \rangle$ is an increasing function of t, as can be seen in Fig. 3.

The paradox is closely related to the flashing ratchet. If we visualize the capital $X(t)$ as the position of a Brownian particle in a one dimensional lattice, game A, for $\epsilon = 0$, is a discretization of the free diffusion, whereas game B resembles the motion of the particle under the action of the asymmetric sawtooth potential. Fig. 4 shows this spatial representation for game B. When the particle is on a dark site, the bad coin is used and the probability to win is very low, whereas on the white sites the most likely move is to the right.

Recall that in the flashing ratchet the sawtooth potential has a short spatial interval in which the force is negative and a long interval with a positive force. Equivalently, game B uses a bad coin on a "short interval", i.e., on one site of every three on the lattice, and a good coin on a "long interval" corresponding to two consecutive sites which are not multiple of three (see Fig. 4). As in the flashing ratchet, game B rectifies the fluctuations of game A. Suppose that we play the sequence AABBAABB... and that $X(t)$ is a multiple of three immediately after two instances of game B. Then we play game A twice, which can drive the capital back to $X(t)$ or to $X(t) \pm 2$. In the latter case, the next turn is for game B with a capital that is not a multiple of three, which means a good chance of winning. That is, game B rectifies the fluctuations that occurred in the two turns of game A. The rectification is not as neat as in the low temperature flashing ratchet, but enough to cause the paradox.

There is a more rigorous way of associating a potential to a gambling game by using a master equation [20]. However, it provides a similar picture of game B, as
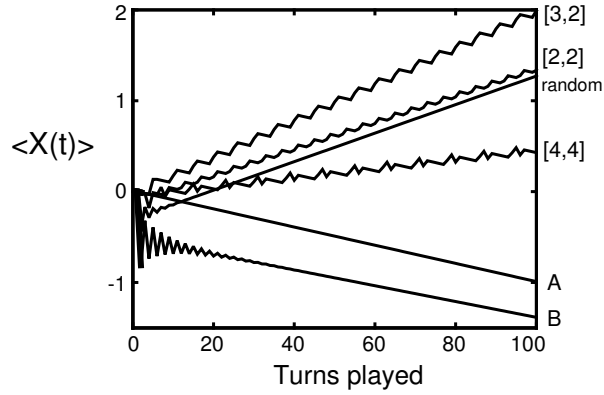
Figure 3: Average capital as a function of the number of turns for game A, B and their periodic and random combinations. $\epsilon = 0.005$ and $[a, b]$ stands for periodic sequences where A (B) is played a (b) consecutive turns.
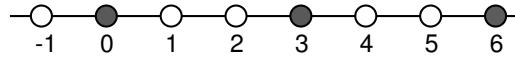

Figure 4: A random walk picture of game B.

a random walk that is nonsymmetric under inversion of the spatial coordinate and capable of rectifying fluctuations.

3.2   Reorganization of trends

We will now present an alternative intuitive explanation of the paradox, showing that game A boosts the effect of the good coin in B, giving the overall game a winning tendency.

Game B is played with good and bad coins. Therefore, for this game the probability to win in the t-th turn can be calculated as

$$p_{\text{win}}(t) = \pi_0(t)p_{\text{bad}} + (1 - \pi_0(t))p_{\text{good}} \tag{1}$$

where $\pi_0(t)$ is the probability of $X(t)$ being a multiple of 3 (i.e. of using the bad coin). To calculate the value of $\pi_0(t)$, we make use of the theory of Markov chains. One can define the Markov process

$$Y(t) \equiv X(t) \mod 3 \tag{2}$$

taking on only three possible values or states, $Y(t) = 0, 1, 2$. Let $\pi_0(t), \pi_1(t), \pi_2(t)$ be the probability of states 0, 1, and 2 at the t-th turn, respectively. This probability distribution obeys the following evolution equation:

$$
\begin{pmatrix} \pi_0(t+1) \\ \pi_1(t+1) \\ \pi_2(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 - p_{\text{good}} & p_{\text{good}} \\ p_{\text{bad}} & 0 & 1 - p_{\text{good}} \\ 1 - p_{\text{bad}} & p_{\text{good}} & 0 \end{pmatrix} \begin{pmatrix} \pi_0(t) \\ \pi_1(t) \\ \pi_2(t) \end{pmatrix}. \quad (3)
$$

After a small number of turns of game B, $\pi_0(t)$ reaches the following stationary value, which is invariant under the transformation given by Eq. (3). The stationary value for π_0 reads

$$
\pi_{0B}^{\text{st}} = \frac{5}{13} - \frac{440}{2197}\epsilon + O(\epsilon^2) \simeq 0.38 - 0.20\,\epsilon. \quad (4)
$$

Substituting this value in Eq. (1) we obtain the probability of winning for game B for sufficiently large t

$$
p_{\text{winB}} = \frac{1}{2} - \frac{147}{169}\epsilon + O(\epsilon^2) \quad (5)
$$

which is less than $1/2$ for $\epsilon > 0$. This proves that game B is losing for $\epsilon > 0$, as shown in Fig. 3.

The paradox arises when game A comes into play. Game A is always played with the same coin, regardless of the value of the capital $X(t)$, and therefore makes the probability of states 0, 1 and 2 tend to a uniform distribution. Thus, game A makes $\pi_0(t)$ tend to $1/3$. Since $1/3 < \pi_{0B}^{\text{st}}$, the effect of game A is to decrease the probability of playing the bad coin when game B is played.

More precisely, when games A and B are chosen at random, the probability of using the bad coin decreases to

$$
\pi_0' = \frac{245}{709} - \frac{48880}{502681}\epsilon + O(\epsilon^2) \simeq 0.35 - 0.10\,\epsilon. \quad (6)
$$

The probability of winning in this randomized combination of games A and B is

$$
p_{\text{win}}' = \pi_0' \frac{p_{\text{bad}} + p_A}{2} + (1 - \pi_0')\frac{p_{\text{good}} + p_A}{2} = \frac{727}{1418} - \frac{486795}{502681}\epsilon + O(\epsilon^2) \quad (7)
$$

which is greater than $1/2$ for a sufficiently small ϵ. This is the mechanism behind the paradox: although the coin in game A is also a bad coin, it increases the probability of playing the good coin in B enough to make the combination win.

Periodic sequences can also be studied as Markov chains and their probability of winning in a whole period can be easily computed. Finally, the slopes of the curves in Fig. 3 can be calculated as $\langle X(t+1) \rangle - \langle X(t) \rangle = 2p_{\text{win}} - 1$.

3.3 Capital-independent games

The modulo rule in game B is quite natural in the original representation of the games as a Brownian ratchet. However, the rule may not suit some applications of the paradox to biology, biophysics, population genetics, evolution, and economics. Thus, it would be desirable to devise new paradoxical games based on rules independent of the capital. Parrondo, Harmer and Abbott introduced such a game in Ref. [21].

In the new version, game A remains the same as before, but a game B', which depends on the history of wins and losses of the player, is introduced. Game B' is played with four coins B'_1, B'_2, B'_3, B'_4 following history-based rules explained in table 1.

Before last $t-2$	Last $t-1$	Coin at t	Prob. of win at t	Prob. of loss at t
Loss	Loss	B'_1	p_1	$1 - p_1$
Loss	Win	B'_2	p_2	$1 - p_2$
Win	Loss	B'_3	p_3	$1 - p_3$
Win	Win	B'_4	p_4	$1 - p_4$

Table 1: History-based rules for game B'

The paradox reappears, for instance, when setting $p_1 = 9/10 - \epsilon$, $p_2 = p_3 = 1/4 - \epsilon$, and $p_4 = 7/10 - \epsilon$. With these numbers and for ϵ small and positive, B' is a losing game, while either a random or a periodic alternation of A and B' produces a winning result. Fig. 5 shows a theoretical computation of the average capital for these history-dependent paradoxical games.

The paradox is reproduced because there are bad coins in game B' which are played more often than in a completely random game, i.e., a quarter of the time. For the above choices of p_i, $i = 1, 2, 3, 4$, the bad coins are B'_2 and B'_3. The other two coins, B'_1 and B'_4, are good coins.

Due to the fact that game B' rules depend on the history of wins and losses, the capital $X(t)$ is no longer a Markovian process. However, the random vector

$$Y(t) = \left(\begin{array}{c} X(t) - X(t-1) \\ X(t-1) - X(t-2) \end{array} \right) \tag{8}$$

can take on four different values and is indeed a Markov chain. The transition probabilities are again easily obtained from the rules of game B' and an analytical solution can be obtained (see [21] for details).
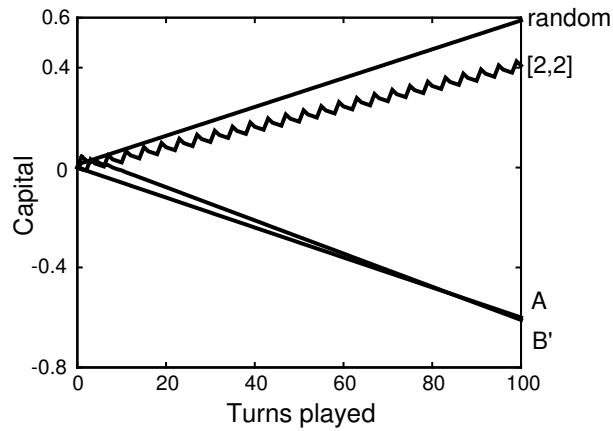
Figure 5: Average capital as a function of the number of turns in the capital independent games, for $\epsilon = 0.003$.

We see that the mechanism that we have called a reorganization of trends can be used to extend the paradox to other gambling games. It is also noteworthy that the price we must pay to eliminate the dependence on the capital in the original paradox is to consider history-dependent rules, i.e., games where the capital is no longer Markovian.

4 Collective games

In this section we analyze three different versions of paradoxical games played by a large number of individuals. The three share the feature that it can sometimes be better for the players to sacrifice short term benefits for higher returns in the future.

4.1 Capital redistribution brings wealth.

Reorganization of trends tells us that the essential role of game A in the paradox is to randomize the capital and make its distribution more uniform. Toral has found that a redistribution of the capital in an ensemble of players has the same effect [11].

In the new paradoxical games introduced by Toral in [11], there are N players and one of them is randomly selected in each turn. With probability 1/2, the selected individual plays game B against the casino; and with probability 1/2 he plays a new game A′ which consists of giving a unit of his capital to another randomly
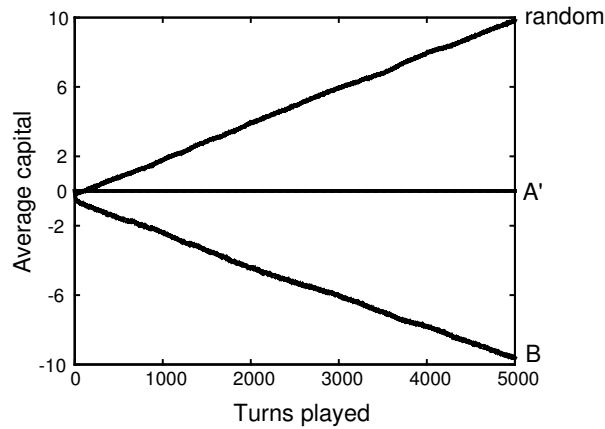
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Figure 6: Average capital per player in game B, game A′ (redistribution of capital), and the random combination, with $\epsilon = 0.01$. The plot depicts the average capital per player as a function of time from a simulation of 10 players and 500 realizations.

chosen player in the ensemble. That is, game B is played in half of the turns and game A′, which is nothing but a redistribution of the total capital, in the other half. The latter obviously does not change the total capital. On the other hand, game B is a losing game and, therefore, playing B without redistribution yields systematic loses. The striking result is that the redistribution of capital turns the losing game into a winning one, actually increasing the total capital available. Thus, the redistribution of capital turns out to be beneficial for everybody. This effect is shown in Fig. 6 where the average total capital in a simulation with 10 players and 500 realizations is depicted for games B and A′, and for their random combination. It is remarkable that the effect is still present when the capital is required to flow from richer to poorer players (see [11] for details).

The explanation to this phenomenon follows the same lines as in the original paradox.

4.2 The voting paradox

Consider now a set of N players who play game A or B against a casino. In each turn, *all* of them play the *same* game. Therefore, they have to make a collective decision, choosing between game A or B in each turn. We will firstly use a *majority rule* (MR) to select the game, that is, the game which receives more votes is played

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by all the players simultaneously.

The interesting feature of this system is that, if the number of players is sufficiently large, it is better for them to vote at random than to vote according to their own benefit in one turn [13]. Voting at random yields a winning tendency while voting for the game that gives the player the highest average return leads to a steady loss, as can be seen in Fig. 7. We will see that, with the MR, selfish voting selects game B most of the time, causing a systematic decrease of the total capital.

In order to explain this behavior, we will focus again on the evolution of $\pi_0(t)$, the fraction of players whose capital is a multiple of three. The selection of the game by voting can be rephrased in terms of $\pi_0(t)$. Every player votes for the game which offers him the higher probability of winning according to his own state. Then, every player whose capital is a multiple of three will vote for game A in order to avoid the bad coin in B. That accounts for a fraction $\pi_0(t)$ of the votes. The remaining fraction $1 - \pi_0(t)$ of the players will vote for game B to play with the good coin. Since the MR establishes that the game which receives more votes is selected, game A will be played if $\pi_0(t) > 1/2$. Conversely, the whole set of players will play game B when $\pi_0(t)$ is below $1/2$.

On the other hand, as we have seen in section 3.2, playing game B makes $\pi_0(t)$ tend to a stationary value given by Eq. (4), namely, $\pi_{0B}^{st} \simeq 0.38 - 0.2\epsilon < 1/2$ for $\epsilon > 0$, whereas playing game A makes π_0 tend to $1/3$. This is still valid for the present model, since the N players only interact when they make the collective decision, otherwise they are completely independent.

If $\pi_0(t) > 1/2$, then the ensemble of players will select game A. The fraction $\pi_0(t)$ will decrease until it crosses this critical value $1/2$. At that turn, B is the selected game and it will remain so as long as π_0 does not exceed $1/2$. However, this can never happen, since game B drives π_0 closer and closer to π_{0B}^{st} which is below $1/2$. Hence, after a number of turns, the system gets trapped playing game B forever with π_0 asymptotically approaching π_{0B}^{st}. But, since ϵ is positive, game B is a losing game (c.f. section 3.2) and, therefore, the MR yields systematic losses, as can be seen in Fig. 7. We have also plotted in Fig. 8 the fraction $\pi_0(t)$, to check that, once $\pi_0(t)$ crosses $1/2$, game B is always chosen and $\pi_0(t)$ approaches π_{0B}^{st}, staying far below $1/2$.

On the other hand, if, instead of using the MR, we select the game at random or following a periodic sequence, game A will be chosen even though $\pi_0 < 1/2$. This is a bad choice for the majority of the players, since playing B would make them toss the good coin. That is, the random or periodic selection will contradict from time to time the will of the majority. Nevertheless, choosing the game at random keeps π_0 away from π_{0B}^{st}, as shown in Fig. 8, i.e., in a region where game B is winning ($\pi_0 < \pi_{0B}^{st}$). Therefore, the random choice yields systematic gains, as shown in Fig. 7.
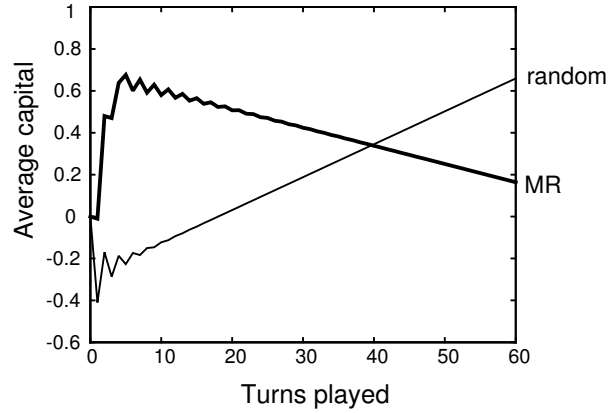
Figure 7: Average capital per player for the random and the voting (MR) choice, for $\epsilon = 0.005$ and an infinite number of players.

It is worth noting that choosing the game at random is exactly the same as if every player voted at random. Therefore, the players get a winning tendency when they vote at random whereas they lose their capital when they vote according to their own benefit in each run.

4.3 The risks of short-range optimization

Yet another "losing now to win later" effect can be observed in the paradoxical games. As in the previous example, we consider a large set of players, but now only a randomly selected fraction γ of them play the game in each turn. Suppose we know the capital of every player so we can compute which game, A or B, will give the larger average payoff in the next turn. Again, and even more strikingly, selecting the "most favorable game" results in systematic losses whereas choosing the game at random or following a periodic sequence steadily increases the average capital [12].

The knowledge of the capital of every player allows us to choose the game with the highest average payoff in the next turn, since this optimal game can easily be obtained from the fraction $\pi_0(t)$ of players whose capital is a multiple of three. These individuals will play the bad coin if game B is chosen and the remaining fraction $1 - \pi_0(t)$ will play the good coin. Hence, the probability of winning for game B reads

$$p_{\text{winB}} = \pi_0 p_{\text{bad}} + (1 - \pi_0)p_{\text{good}}. \tag{9}$$
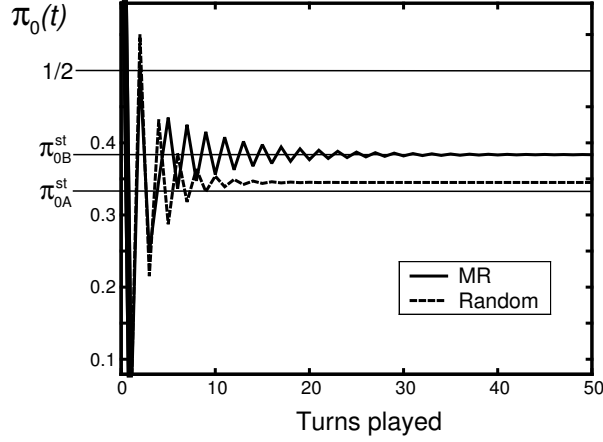
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Figure 8: The fraction of players, $\pi_0(t)$, with capital multiple of three under the MR and the random choice ($\epsilon = 0.005$ and $N = \infty$). The horizontal lines indicate the threshold value for the MR choice (1/2), and the stationary values for games A and B, π_{0A}^{st} and π_{0A}^{st}, respectively.

In case game A is selected, the probability to win is $p_{winA} = p_A = 1/2 - \epsilon$ for all time t. Therefore, to choose the game with the larger payoff $\langle X(t+1) \rangle - \langle X(t) \rangle = 2p_{win} - 1$ in every turn t, we must

$$
\begin{aligned}
&\text{play A} \quad \text{if} \quad p_{winA} \geq p_{winB}(\pi_0) \\
&\text{play B} \quad \text{if} \quad p_{winA} < p_{winB}(\pi_0)
\end{aligned}
\tag{10}
$$

or equivalently

$$
\begin{aligned}
&\text{play A} \quad \text{if} \quad \pi_0(t) \geq \pi_{0c} \\
&\text{play B} \quad \text{if} \quad \pi_0(t) < \pi_{0c}
\end{aligned}
\tag{11}
$$

with $\pi_{0c} \equiv (p_A - p_{good})/(p_A - p_{bad}) = 5/13$. We will call this way of selecting the game *the short-range (SR) optimal strategy*. We will also consider that the game is selected following either a random or periodic sequence. These are both *blind* strategies, since they do not make any use of the information about the state of the system. However, and surprisingly enough, they turn out to be much better that the SR optimal strategy, as shown in Fig. 9.

Notice that (11) is similar to the way the game is selected by the MR of the previous section, but replacing $1/2$ by the new critical value $\pi_{0c} = 5/13$. Therefore, the explanation of this model goes quite along the same lines as for the voting
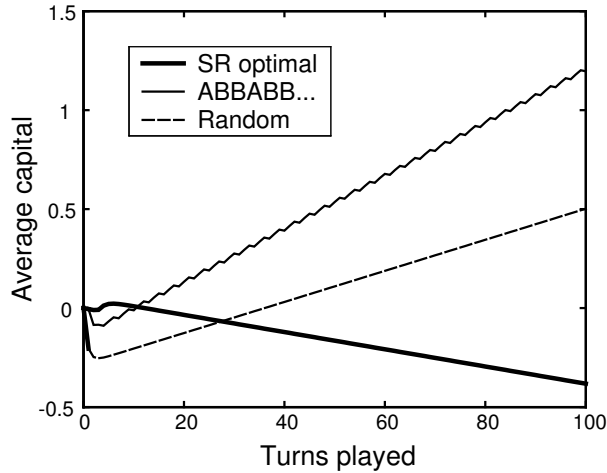
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Figure 9: Average capital in function of time for the three different strategies explained in the text, with $N = \infty$, $\gamma = 0.5$, and $\epsilon = 0.005$.

paradox, although with some differences. Unlike $1/2$, π_{0c} equals the stationary value of $\pi_0(t)$ for game B when $\epsilon = 0$. As in the voting paradox, game A drives π_0 below π_{0c} because game A makes π_0 tend to $1/3$. If $\pi_0(t) < \pi_{0c}$, then game B is played, but $\pi_0(t+1)$ will be still below π_{0c} only for γ sufficiently small. For example, if $\gamma = 1/2$ and $\epsilon = 0$, game B is chosen forty times in a row before switching back to game A, making π_0 become approximately equal to π_{0B}^{st} at almost every turn. This behavior is shown in Fig. 10. As long as π_0 is close to π_{0B}^{st}, the average capital remains approximately constant, as shown in Fig. 11.

In contrast, the periodic and random strategies choose game A with $\pi_0 < \pi_{0c}$. Although this does not produce earnings in that turn, it keeps π_0 away from π_{0B}^{st}. When game B is chosen again, it has a large expected payoff since π_0 is not close to π_{0B}^{st}. By keeping π_0 not too close to π_{0B}^{st}, the blind strategies perform better than the short-range optimal prescription, as can be seen in Fig. 11.

The introduction of $\epsilon > 0$ has two effects. First of all, it makes π_{0B}^{st} decrease by a small amount, as indicated in Eq. (4). This makes it even more difficult for the SR strategy to choose game A, and after a few runs game B is always selected. Since game B is now a losing game, the SR optimal strategy is also losing whereas periodic and random strategies keep their winning tendency, as can be seen in Fig. 9.

To summarize, the SR optimal strategy chooses B most of the times, since it is the game which gives the highest returns in each turn. However, this choice drives $\pi_0(t)$ to a region in which B is no longer a winning game. On the other hand, the
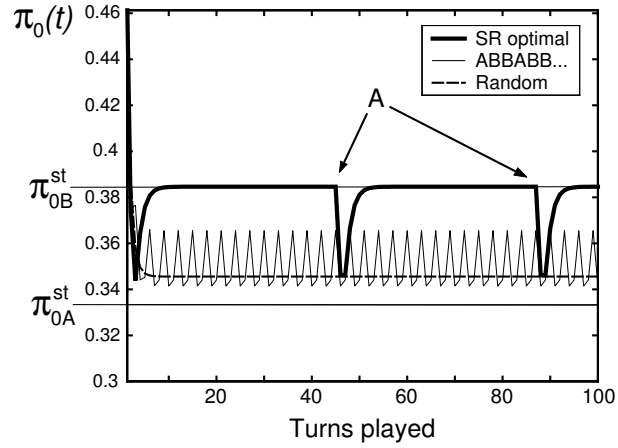
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Figure 10: $\pi_0(t)$ for $\epsilon = 0$, $N = \infty$, and $\gamma = 0.5$. The horizontal lines show the stationary values for game A and game B (which coincides with the critical fraction π_{0c} for the SR optimal strategy).
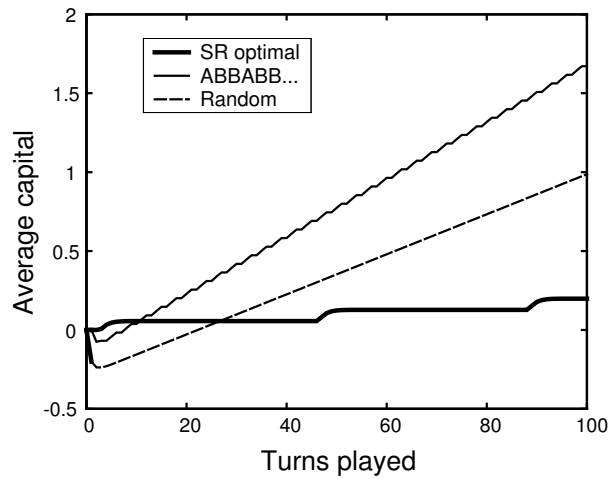

Figure 11: Average capital as a function of time, for the three different strategies explained in the text, $\epsilon = 0$, $N = \infty$ and $\gamma = 0.5$.

random strategy from time to time sacrifices the short term returns by selecting game A, but this choice keeps the system in a "productive region". We could say that the SR optimal strategy is "killing the goose that laid the golden eggs", an effect that is also present in simple deterministic systems [12].

5 Patterns

The original paradox illustrates how simple switching between dynamics, each of which produces an "undesirable" state, may lead to a "desirable" outcome. In this section we show that this idea can be generalized to other topics in the context of spatially extended systems. Spatiotemporal pattern formation in nonequilibrium extended systems plays a relevant role in a number of phenomena, and in the past few decades there has been continued progress in the understanding of different mechanisms that lead to such patterns [22]. Following the ideas of the games, we show herein that the alternation of dynamics, neither of which exhibits patterns, constitutes a striking mechanism for pattern formation [14, 15, 16].

5.1 Recipe for pattern formation

To illustrate the mechanism, we consider a simple family of models that exhibit patterns. In general, the overdamped Langevin dynamics for a scalar field $\varphi(\mathbf{r}, t)$ reads:

$$\dot{\varphi}(\mathbf{r}, t) = -V'\left(\varphi(\mathbf{r}, t)\right) + \mathcal{L}\varphi(\mathbf{r}, t) + \xi(\mathbf{r}, t). \tag{12}$$

The temporal evolution of the field is driven by a local force that can be derived from a local potential, $V(\varphi)$, by its coupling with other locations, indicated by the operator $\mathcal{L}$, and by fluctuations (for example, thermal fluctuations) modelled by the random term $\xi(\mathbf{r}, t)$. We assume that $\xi(\mathbf{r}, t)$ is Gaussian, has zero mean value, and correlation function

$$\langle \xi(\mathbf{r}, t)\xi(\mathbf{r}', t') \rangle = \sigma^2 \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t'). \tag{13}$$

A system such as (12) must satisfy two requirements for pattern formation: the local potential must have at least two stable equilibrium points, and the coupling must induce a morphological instability [22], i.e., $|\mathbf{k}| = 0$ can not be the most unstable Fourier mode. A paradigmatic example is the Swift-Hohenberg (SH) model [23], a phenomenological model for the Rayleigh-Benard system near the convection threshold.

For the SH model the coupling term reads $\mathcal{L}_{\mathrm{SH}} \equiv -\left(1 + \nabla^2\right)^2$. Note that this coupling operator determines a morphological instability with $|\mathbf{k}^*| = 1$ as the most

unstable Fourier mode. Throughout this section we will consider the coupling in Eq. (12) to be $\mathcal{L}_{\mathrm{SH}}$.

If $V(\varphi)$ is monostable no patterns appear, and the steady state of the system is spatially homogeneous. The homogeneous state is then determined by the equilibrium point, $\widetilde{\varphi}$, of the *effective* local potential

$$\widetilde{V}(\varphi) = V(\varphi) + \frac{\varphi^2}{2}. \tag{14}$$

The term $\varphi^2/2$ arises from the "1" in the coupling term acting on the field. Thus, $\widetilde{\varphi}$ is the solution of

$$V'(\widetilde{\varphi}) + \widetilde{\varphi} = 0. \tag{15}$$

Note that although $V(\varphi)$ is monostable, $\widetilde{V}(\varphi)$ may not be, and one may wonder about the possibility of generating a pattern despite the fact that the local potential has only one equilibrium point. However, this does not occur. Considering small fluctuations around the homogeneous state, $\varphi = \widetilde{\varphi} + \delta$, and linearizing Eq. (12), leads to the following evolution equation for the Fourier component (indicated by a hat) of the field for the most unstable modes $\mathbf{k}^*$:

$$\dot{\widehat{\delta}}(\mathbf{k}^*, t) = -V''(\widetilde{\varphi})\widehat{\delta}(\mathbf{k}^*, t). \tag{16}$$

This leads to unstable behavior only if $V''(\widetilde{\varphi}) < 0$. Since $V(\varphi)$ has only one equilibrium point, it follows that $V''(\widetilde{\varphi}) > 0$ and thus no pattern arises even if $\widetilde{V}(\varphi)$ is not monostable. Moreover, it may happen that $V(\varphi)$ and $\widetilde{V}(\varphi)$ are *not* monostable and yet no structure develops because $V''(\widetilde{\varphi}) > 0$. Hence we arrive at the following conditions:

$$\text{if } V'(\widetilde{\varphi}) + \widetilde{\varphi} = 0 \begin{cases} \text{ but } V''(\widetilde{\varphi}) > 0, & \text{then no pattern develops.} \\ \\ \text{ and } V''(\widetilde{\varphi}) < 0, & \text{then a pattern develops.} \end{cases} \tag{17}$$

5.2 Global Switching

Consider now a *global* switching mechanism between two local potentials $V_1(\varphi)$ and $V_2(\varphi)$:

$$\dot{\varphi}(\mathbf{r}, t) = -\Lambda(t)V_1'\left(\varphi(\mathbf{r}, t)\right) - (1 - \Lambda(t))\,V_2'\left(\varphi(\mathbf{r}, t)\right) + \mathcal{L}\varphi(\mathbf{r}, t) + \xi(\mathbf{r}, t). \tag{18}$$

Here $\Lambda(t)$ is a dichotomous function of time that takes on the values 0 and 1. In this way, either $V_1(\varphi)$ or $V_2(\varphi)$ acts on the system at *every* site at a given time. It is easy to check that (18) can be rewritten as

$$\dot{\varphi}(\mathbf{r}, t) = -V_+'\left(\varphi(\mathbf{r}, t)\right) - \mu(t)V_-'\left(\varphi(\mathbf{r}, t)\right) + \mathcal{L}\varphi(\mathbf{r}, t) + \xi(\mathbf{r}, t), \tag{19}$$

where $\mu(t) = 2\Lambda(t) - 1 = \pm 1$, and

$$V_\pm(\varphi) \equiv \frac{V_1(\varphi) \pm V_2(\varphi)}{2}. \tag{20}$$

Suppose that $V_{1,2}(\varphi)$ and $\widetilde{V}_{1,2}(\varphi)$ are *monostable* potentials. It is then clear, according to conditions (17), that neither of the two dynamics by itself will lead to patterns. Using the language of the games, at any time and at all sites we are "playing" a "losing" game: no pattern formation. However, we will show that the nonequilibrium process of alternation in time, either periodically or randomly, may lead to a "winning outcome": different kinds of oscillatory and stationary patterns. The reason is now the stabilization of transient states mentioned in the introduction.

Let t_s denote the average time that the system spends in each dynamics. We then expect that if $t_s \to \infty$, that is, if switching is slow, every site will reach the equilibrium point $\widetilde{\varphi}_i$ appropriate to the potential $V_i(\varphi)$ that drives the system. Therefore, the field will oscillate between homogeneous structures. However if the switching process is sufficiently fast (later we will state the condition quantitatively), the fast variable $\mu(t)$ can be replaced by its average value, $\mu(t) \sim \langle \mu(t) \rangle = 0$. Therefore, in that limit the system is *effectively* driven by the potential $V_+(\varphi)$. We stress that, although $V_{1,2}(\varphi)$ are monostable and satisfy the condition (17) associated with *no* pattern formation, $V_+(\varphi)$ may in general satisfy either condition. In particular, if $V_{1,2}(\varphi)$ are such that

$$V_i'(\widetilde{\varphi}_i) + \widetilde{\varphi}_i = 0 \quad \text{and} \quad V_i''(\widetilde{\varphi}_i) > 0, \tag{21}$$

$$V_+'(\widetilde{\varphi}_+) + \widetilde{\varphi}_+ = 0 \quad \text{and} \quad V_+''(\widetilde{\varphi}_+) < 0, \tag{22}$$

pattern formation will occur due to the *global* temporal alternation of two dynamics neither of which alone leads to patterns.

5.3 Relaxation transients between dynamics

Given any particular choice of $V_{1,2}(\varphi)$ satisfying Eqs. (21) and (22), the formation of spatial structures can be understood in terms of the ratio r of the two characteristic times of the system: the time that the system spends in each dynamics, t_s, and the relaxation time to equilibrium states, t_r:

$$r = \frac{t_s}{t_r}. \tag{23}$$

The time t_r is the smaller of $t_{1\to2}$ and $t_{2\to1}$, where $t_{i\to j}$ is the relaxation time, *under the action of V_j*, of the homogeneous state associated with V_i. We can estimate

$t_{i \to j}$ by focusing only on the $|\mathbf{k}| = 0$ mode and assuming that, when the potential switches from V_i to V_j, the mode amplitude behaves as a Brownian particle initially equilibrated in the effective local potential $\widetilde{V}_i(\varphi)$. When the local potential is switched, this point, which up to that moment was stable, becomes unstable. The relaxation time to the new homogeneous state associated with V_j is the time that it takes the Brownian particle to roll down the potential hill to the new equilibrium point [24]:

$$ t_{i \to j} = \frac{2}{\sigma^2} \int_{\tilde{\varphi}_i}^{\tilde{\varphi}_j} \mathrm{d}y \exp\left(\frac{2}{\sigma^2}\widetilde{V}_j(y)\right) \int_{\tilde{\varphi}_i}^{y} \mathrm{d}z \exp\left(-\frac{2}{\sigma^2}\widetilde{V}_j(z)\right). \qquad (24) $$

On the other hand, the time that the system spends in one of the two dynamics, t_s, reads as follows. If the dichotomous switching is periodic, t_s is clearly the semi-period of the signal, $t_s = T/2$. If the switching is random, we take $\Lambda(t)$ to be a dichotomous exponentially correlated random variable with correlation time τ. The correlation function of the associated random dichotomous variable $\mu(t)$ then is

$$ \langle \mu(t)\mu(t') \rangle = e^{-|t-t'|/\tau}. \qquad (25) $$

The time that the system spends in each dynamics on average is then $t_s = 2\tau$.

If $r \gg 1$ the system will alternate between homogeneous states and if $r \ll 1$ a stationary pattern will be obtained. The case $r \sim 1$ is the most striking: when the switching is periodic, a resonance phenomenon between the two characteristic times of the system may produce oscillatory patterns. These patterns only occur under periodic switching, that is, random switching even with a ratio $r \sim 1$ does not produce *sustained* oscillatory patterns [15].

5.4 A Particular Case

Let us now focus on the following particular family of local potentials that satisfy the conditions (21) and (22):

$$ V_{1,2}(\varphi) = \frac{\varphi^4}{4} \pm \frac{\varphi^3}{3} - \frac{\varphi^2}{2} \mp \varphi. \qquad (26) $$

Then the potentials $V_\pm(\varphi)$ are

$$ V_\pm(\varphi) = a\frac{\varphi^4}{4} + a_\mp\frac{\varphi^3}{3} - a\frac{\varphi^2}{2} - a_\mp\varphi, \qquad (27) $$

where $a_+ = 1$ and $a_- = 0$. In Fig. 12 we show the effective monostable potentials $\widetilde{V}_{1,2}(\varphi)$.
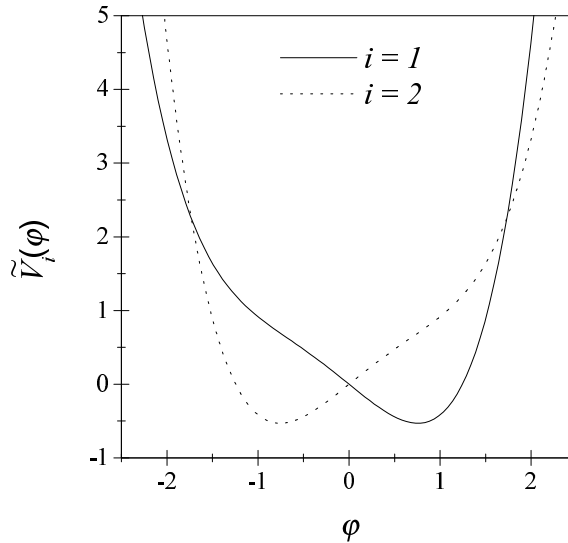
Figure 12: Effective local potentials $\widetilde{V}_1(\varphi)$ (solid curve) and $\widetilde{V}_2(\varphi)$ (dotted curve).

We first compute the relaxation time t_r. Using Eq. (24) with $\sigma^2 = 10^{-2}$ we obtain $t_r \simeq 2.2$. This value is in agreement with that found in numerical experiments, $t_r = 2.49 \pm 10^{-2}$.

We now show the results of one-dimensional (1-d) simulations. The values of the relevant parameters are $\Delta t = 10^{-3}$, $\Delta x = 0.5$, $L = 64$, and $\sigma = 10^{-2}$. We expect the typical wavelength of the pattern to be $\lambda = 2\pi/|\mathbf{k}^*| \simeq 2\pi$ and the aspect ratio $L/\lambda \sim 10$, that is, when a pattern develops we expect to find 10 wavelengths inside the lattice. In order to avoid possible instabilities arising from boundary effects we implement periodic boundary conditions. The initial condition is taken to be random according to a Gaussian distribution. As for the effect of the additive fluctuations in the dynamics, only if the initial condition were chosen to be uniform, $\varphi(\mathbf{r}, 0) = $ const. for all $\mathbf{r}$, are they relevant since in all other cases small fluctuations do not play a significant dynamical role. Clearly, a uniform initial condition does not produce patterns in the deterministic problem regardless of the value of the control parameter r.

In Fig. 13 we show the results of 1-d simulations with random switching. We present a density plot of the field as a function of space and time for $r = 4.5$ (left panel) and $r = 0.045$ (central panel). In the first case a clear alternation between homogeneous states is obtained, and in the second we see the formation of a stationary pattern.
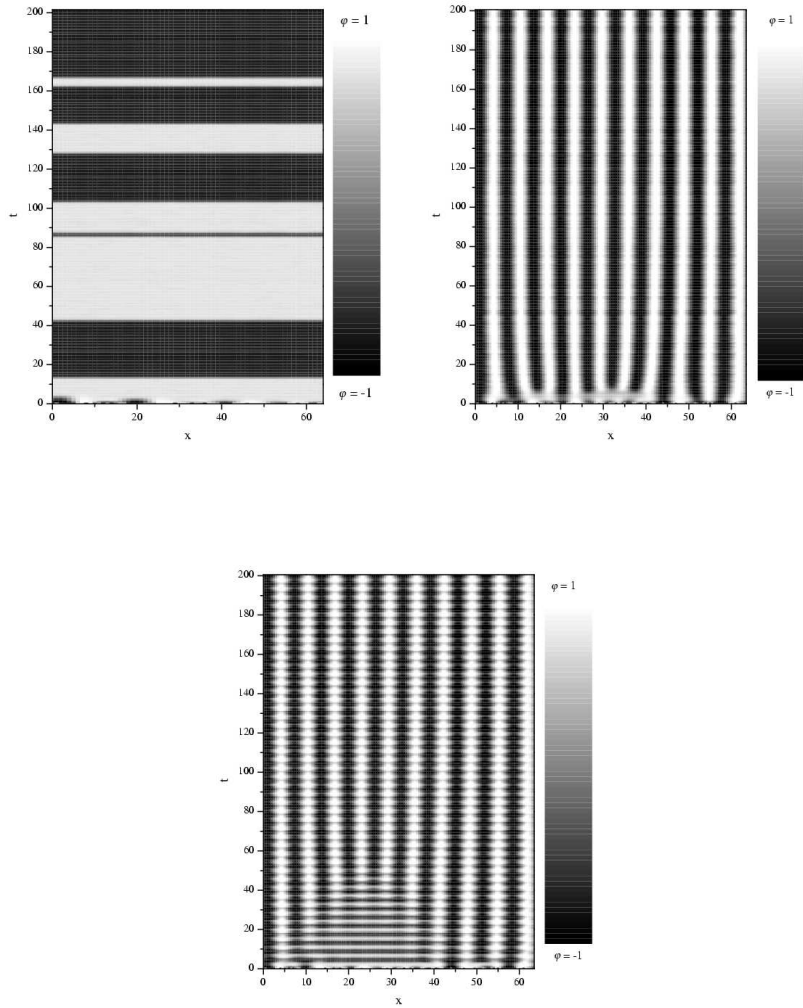
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Figure 13: Spatiotemporal density plot of the field with slow random switching (left), fast random switching (center) and periodic resonant switching (right): $r = 4.5$, $r = 0.045$, and $r = 1.15$ respectively. A clear alternation of homogeneous states is observed in the first case, while stationary (center) and oscillatory (right) patterns develop in the other cases.

Deterministic periodic switching leads to results similar to those of random switching when $r \gg 1$ (alternation of homogeneous states) and $r \ll 1$ (stationary patterns). In addition, when $r \simeq 1$ an oscillatory pattern develops. The right panel of Fig. 13 shows, again by means of a density plot of the field as a function of space and time, the oscillatory structure that arises when $r = 1.15$. Such a spatiotemporal structure resembles the so-called *oscillons* found in granular materials [25]. It is also worth pointing out that for 2-d systems different symmetries determine the spatial arrangements and shapes of patterns [14, 15].

6 Conclusions

We have presented the original Parrondo's paradox and several examples showing how the basic mechanisms underlying the paradox can yield other counter-intuitive phenomena in collective games and in spatially extended systems.

The first mechanism, the ratchet effect, occurs when fluctuations can help to surmount a potential barrier or a "losing streak". These fluctuation can either come from another losing game, such as in the original paradox, from a redistribution of the capital, such as in Toral's collective games, or from a purely diffusive motion, such as in the flashing ratchet.

A second mechanism is the reorganization of trends, which occurs when game A reinforces a positive trend already present in game B. The same mechanism can be observed in the games with capital independent rules and it helps to understand the counter-intuitive behavior of the collective games presented in section 4.2 and 4.3, where random choices perform better than the choice preferred by the majority or the one optimizing short term returns. These models also prompt the question of how information can be used to design a strategy. It is a relevant question for control theory and also for statistical mechanics, since the paradox is a purely collective effect that goes away for a single player, i.e., the SR optimal strategy and the MR choices perform much better than the random or periodic ones.

Finally, the outcome of an alternation of dynamics can always be interpreted as a stabilization of transient states. This interpretation allowed us to extend the basic message of the paradox to pattern formation in spatially extended systems. Drawing parallels with the games, we have shown how the global alternation of two dynamics, each of which leads to a homogeneous steady state ("losing" dynamics), can produce stationary or oscillatory patterns ("winning" dynamics) upon alternation. The appearance of spatial or spatiotemporal patterns depends on the ratio r of the alternation time to the relaxation time of the system in the slower of the two dynamics. Random alternation leads to stationary spatial patterns, while periodic alternation may lead to stationary or oscillatory spatial patterns [14, 15, 16]. The

alternation mechanism has been presented for certain classes of models based on the Swift-Hohenberg equation. One can envision many other situations in which global alternation between homogeneous or even chaotic dynamics may lead to spatiotemporal pattern formation [26]. Moreover, the generalization to other situations of the underlying idea, namely, that the averaged dynamics may behave differently than its dynamical components, is straightforward and may lead to further striking results in the behavior of dynamical systems [27].

One of these extensions concerns quantum systems. Lee *et al.* have devised a toy model in which the alternation of two decoherence dynamics can significantly decrease the decoherence rate of each separate dynamics [28]. Also in the quantum domain, the paradox has received some attention and has been reproduced in the contexts of quantum lattice gases [29], quantum games [30, 31], and quantum algorithms [32].

To finish this partial account of the existing literature on the paradox, we mention the work by Arena *et al* [33], who analyze the performance of the games using chaotic instead of random sequences of choices; that of Chang and Tsong [34], who study the hidden coupling between the two games in the paradox and present several extensions even for deterministic dynamics; and the paper by Kocarev and Tasev [35], relating the paradox with Lyapunov exponents and stochastic synchronization.

In summary, Parrondo's paradox has drawn the attention of many researchers to non-trivial phenomena associated with switching between two dynamics. In this paper, we have tried to reveal some of the basic mechanisms that can yield an unexpected behavior when switching between two dynamics, and how these mechanisms work in several versions of the paradox. We believe that the paradox and its extensions are contributing to a deeper understanding of stochastic dynamical systems. In the case of statistical mechanics, switching is in fact a source of non-equilibrium which is ubiquitous in nature, due to day-night or seasonal variations [27]. Nevertheless, it has not been studied in depth until the recent introduction of ratchets and paradoxical games. As the paradox suggests, we will probably see in the future new models and applications confirming that noise and switching, even between equilibrium dynamics, can be a powerful combination to create order and complexity.

Acknowledgements

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