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Inefficiency of voting in Parrondo games

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Abstract

We study a modification of the so-called Parrondo's paradox where a large number of individuals choose the game they want to play by voting. We show that it can be better for the players to vote randomly than to vote according to their own benefit in one turn. The former yields a winning tendency while the latter results in steady losses. An explanation of this behaviour is given by noting that selfish voting prevents the switching between games that is essential for the total capital to grow. Results for both finite and infinite number of players are presented. It is shown that the extension of the model to the history-dependent Parrondo's paradox also displays the same effect.

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1. Introduction

The dynamics of a flashing ratchet [1–4] can be translated into a counterintuitive phenomenon in gambling games which has recently attracted considerable attention. It is the so-called *Parrondo's paradox* [5–8] consisting of two losing games, A and B, that yield, when alternated, a winning game.

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In game A, a player tosses a coin and makes a bet on the throw. He wins or loses 1 euro depending on whether the coin falls heads or tails. The probability p_1 of winning is $p_1 = \frac{1}{2} - \epsilon$ with $0 \leq \epsilon \ll 1$; so game A is fair when $\epsilon = 0$ and losing when $\epsilon > 0$. By losing, winning, and fair games here we mean that the average capital is a decreasing, increasing, and a constant function of the number of turns, respectively.

The second game—or game B—consists of two coins. The player must throw coin 2 if his capital is not a multiple of three, and coin 3 otherwise. The probability of winning with coin 2 is $p_2 = \frac{3}{4} - \epsilon$ and with coin 3 is $p_3 = \frac{1}{10} - \epsilon$. They are called “good” and “bad” coins respectively. It can be shown that game B is also a losing game if $\epsilon > 0$ and that $\epsilon = 0$ makes B a fair game [7,8]. The rules of both games A and B are depicted in Fig. 1.

Surprisingly, switching between games A and B in a random fashion or following some periodic sequences produces a winning game, for $\epsilon > 0$ sufficiently small, i.e., the average of player earnings grows with the number of turns. Therefore, from two losing games we actually get a winning game [5–7]. This indicates that the alternation of stochastic dynamics can result in a new dynamics, which differs qualitatively from the original ones.

Alternation is either periodic or random in the flashing ratchet [1–4] and in the paradoxical games [5–7]. On the other hand, we have recently studied the case of a *controlled* alternation of games, where information about the state of the system can be used to select the game to be played with the goal of maximising the capital [9]. This problem is trivial for a single player: the best strategy is to select game A when his capital is a multiple of three and B otherwise. This yields higher returns than any periodic or random alternation. Therefore, choosing the game as a function of the current capital presents a considerable advantage with respect to “blind” strategies, i.e., strategies that do not make use of any information about the state of the system, as it is the case of the periodic and random alternation. Analogously, in a flashing ratchet, switching on and off the ratchet potential depending on the location of the Brownian particle allows one to extract energy from a single thermal bath, in apparent contradiction with the second law of thermodynamics [10]. This is nothing

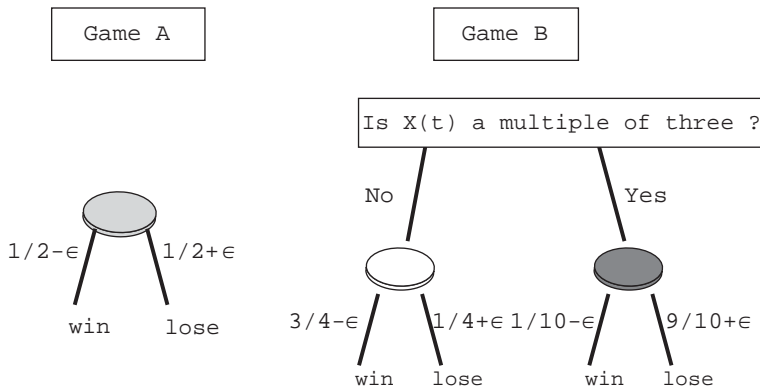


Fig. 1. Rules of the two Parrondo games.

but a Maxwell demon, who operates having at his disposal information about the position of the particle; and it is the acquisition or the subsequent erasure of this information what has an unavoidable entropy cost [11], preventing any violation of the second law.

Whereas a controlled alternation of games is trivial for a single player, interesting and counter-intuitive phenomena can be found in *collective* games. In Ref. [9], we considered a collective version of the original Parrondo's paradox. In this model, the game—A or B—that a large number N of individuals play can be selected at every turn. It turns out that blind strategies are winning whereas a strategy which chooses the game with the highest average return is losing [9].

In the present paper, we extend our investigation of controlled collective games considering a new strategy based on a majority rule, i.e., on voting. This type of rule differs from the one considered in Ref. [9] and has been proved to be relevant in several situations, such as the modelling of public opinion [12,13] or the design of multi-layer neural networks by means of *committee machines* [14,15]. We will show that, in controlled games, the rule is very inefficient: if every player votes for the game that gives him the highest return, then the total capital decreases, whereas blind strategies generate a steady gain. As mentioned above, for a single player, the majority rule does defeat the blind strategies. The inefficiency of voting is consequently a purely collective effect.

In contrast with the short range optimisation strategy in Ref. [9] where the average returns are maximised, the players now choose the game according to their own benefit in the next coin toss and never care about how bad is that decision for the rest of the community. Hence, the inefficiency of the control is now stronger: the short-range optimisation is only worse than the blind strategies if just a random selected set of the players are allowed to play the games at each turn whereas the majority rule is losing no matter how many of the players are allowed to vote and play.

Finally, the same effect can be found for the capital-independent games introduced in Ref. [8], showing that the mechanism underlying the inefficiency of voting is general and can be extended to other systems while retaining its main features.

The paper is organised as follows. In Section 2, we present the model and the counter-intuitive performance of the different strategies. In Section 3, we discuss and provide an intuitive explanation of this behaviour. In Section 4, we analyse how the effect depends on the number of players. In Section 5, we extend these ideas to the capital-independent games introduced in Ref. [8]. Finally, in Section 6 we present our main conclusions.

2. The model

The model consists of a large number N of players. In every turn, they have to choose one of the two original Parrondo games, described in the Introduction and in Fig. 1. Then *every* individual plays the selected game against the casino.

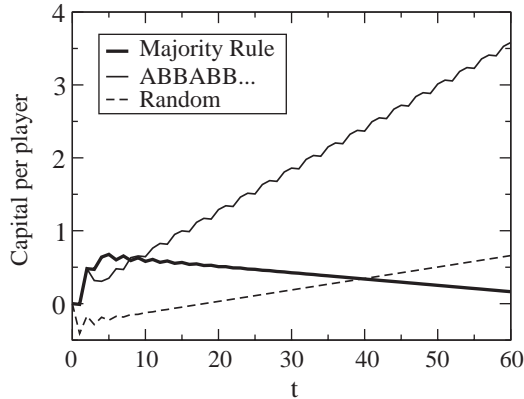


Fig. 2. Evolution of the capital per player in an infinite ensemble for $\varepsilon = 0.005$ and the three strategies discussed in the text.

We will consider three strategies to achieve the collective decision. (a) The *random strategy*, where the game is chosen randomly with equal probability. (b) The *periodic strategy*, where the game is chosen following a given periodic sequence. The sequence that we will use throughout the paper is *ABBABB...* since it is the one giving the highest returns. (c) The *majority rule (MR) strategy*, where every player votes for the game giving her the highest probability of winning, with the game obtaining the most votes being selected.

The model is related to other extensions of the original Parrondo games played by an ensemble of players, such as those considered by Toral [16,17]. However, in our model the only interaction among players can occur when the collective decision is made. Once the game has been selected, each individual plays, in a completely independent way, against the casino. Moreover, in the periodic and random strategies there is no interaction at all among the players, the model being equivalent to the original Parrondo's paradox with a single player.

The MR makes use of the information about the state of the system, whereas the periodic and random strategies are blind, in the sense defined above. One should then expect a better performance of the MR strategy. However, it turns out that, for large N , these blind strategies produce a systematic winning whereas the MR strategy is losing. This is shown in Fig. 2, where the capital per player as a function of the number of turns is depicted for the three strategies and an infinite number of players (see Appendix A for details on how to obtain Fig. 2).

3. Analysis

How many players vote for each game? The key magnitude to answer this question and to explain the system's behaviour is $\pi_0(t)$, the fraction of players whose money is a multiple of three at turn t . This fraction $\pi_0(t)$ of players vote for game A in order to

avoid the bad coin in game B. On the other hand, the remaining fraction $1 - \pi_0(t)$ vote for game B to play with the good coin. Therefore, if $\pi_0(t) \geq 1/2$, there are more votes for game A and, if $\pi_0(t) < 1/2$, then game B is preferred by the majority of the players.

Let us focus now on the behaviour of $\pi_0(t)$ for $\varepsilon = 0$ when playing both games separately. If game A is played a large number of times, $\pi_0(t)$ tends to $1/3$ because the capital is a symmetric and homogenous random walk under the rules of game A. On the other hand, if B is played repeatedly, $\pi_0(t)$ tends to $5/13$. This can be proved by analyzing game B as a Markov chain [7,8]. It is also remarkable that, for $\pi_0(t) = 5/13$, the average return when game B is played is zero.

Fig. 3 represents schematically the evolution of $\pi_0(t)$ under the action of each game, as well as the prescription of the MR strategy explained above. Now we are ready to explain why the MR strategy yields worse results than the periodic and random sequences.

We see that, as long as $\pi_0(t)$ does not exceed $1/2$, the MR strategy chooses game B. However, playing B takes π_0 closer to $5/13$, well below $1/2$, and thus more than half of the players vote for game B again. After a number of runs, the MR strategy gets trapped playing game B forever. Then π_0 asymptotically approaches $5/13$, and as this happens, game B turns into a fair game when $\varepsilon = 0$. As a consequence, the MR will not produce earnings any more, as can be seen in Fig. 4.

The introduction of $\varepsilon > 0$ turns game B into a losing game if played repeatedly. Consequently, the MR strategy becomes a losing one as in Fig. 2. To overcome this losing tendency, the players must sacrifice their short-range profits, not only for the benefit of the whole community but also for their own returns in the future. Hence, some kind of cooperation among the players is needed to prevent them from losing their capital. A similar effect has been found by Toral in another version of collective Parrondo's games. There, sharing the capital among players induces a steady gain [16]. In our case, the striking result is that no complex cooperation is necessary. It is enough that the players agree to vote at random.

We would like to stress the differences with the short-range optimisation considered in Ref. [9]. In that paper, the critical value of $\pi_0(t)$ was $5/13$ instead of $1/2$. As a consequence, the voting strategy is even less efficient than the short-range optimisation. For instance, the short-range optimisation is winning if the whole

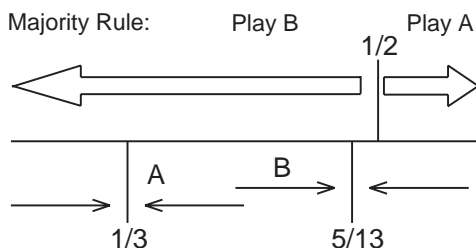


Fig. 3. Schematic representation of the evolution of $\pi_0(t)$ under the action of games A and B. The prescription of the MR is also represented.

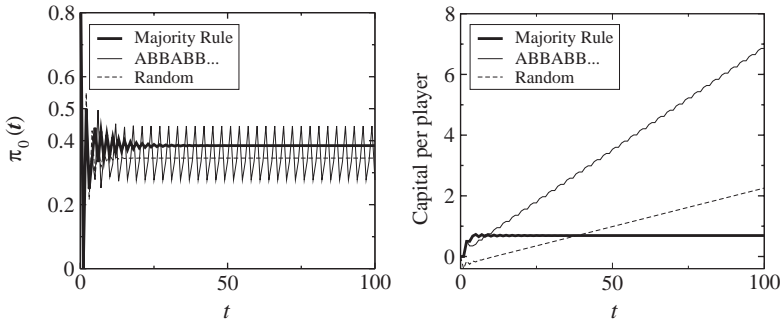


Fig. 4. Evolution of $\pi_0(t)$ (left) and the capital per player (right) for $N = \infty$, $\varepsilon = 0$ for the MR and random strategies. The MR chooses game B when π_0 is below the straight line depicted at $1/2$ and game A otherwise.

ensemble plays in each turn. To achieve the counterintuitive phenomena that blind strategies perform better than the short-range optimisation, in Ref. [9] we were forced to include the following rule: in each turn only a fraction γ of the individuals in the ensemble play the game. For the voting strategy, the inclusion of this rule is not necessary.

4. Finite number of players

In the previous analysis an infinite number of players has been considered. Remarkably, for just one player the MR strategy trivially performs better than any periodic or random sequence, since it completely avoids the use of the bad coin. In this section, we analyse the crossover between the winning behaviour for a small number of players and the losing behaviour when this number is large.

Fig. 5 shows numerical results of the average capital per player for an increasing number of players ranging from 10 to 1000. One can observe that, the larger the number of players, the worse the results for the MR strategy, becoming losing for a number of players between 50 and 100.

The above discussion for an infinite ensemble allows us to give a qualitative explanation. The difference between large and small N is the magnitude of the fluctuations of $\pi_0(t)$ around its expected value. If game B is chosen a large number of times in a row, then the expected value of $\pi_0(t)$ is $5/13$. On the other hand, the MR selects B unless $\pi_0(t)$ is above $1/2$. Therefore, for the MR to select A, fluctuations must be of order $1/2 - 5/13 = 3/26 \simeq 0.115$. For N players, the fraction of players with capital multiple of three, $\pi_0(t)$, will be a random variable following a binomial distribution, at least if B has been played a large number of times in a row. If the expected value of $\pi_0(t)$ is $5/13$, fluctuations of $\pi_0(t)$ around this value are of order $\sqrt{5/13 \times 8/13 \times 1/N}$. Then, fluctuations will allow the MR strategy to choose A if $N \simeq 20$. Far above this value, fluctuations that drive $\pi_0(t)$ above $1/2$ are very rare, and MR chooses B at almost every turn, as can be seen in Fig. 6. On the other hand,

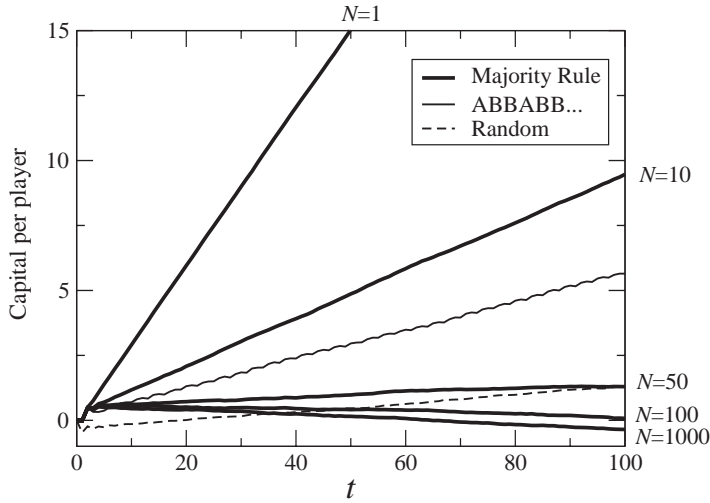


Fig. 5. Simulation results for the average capital per player for $N = 10, 50, 100,$ and 1000 players, $\varepsilon = 0.005$, and the three different strategies. The simulations have been made over a variable number of realizations, ranging from 10 000 realizations for $N = 1$ to 10 realizations for $N = 1000$. Simulations for the random and periodic strategies have been made with $N = 100$ players and averaging over 100 realizations. For these blind strategies, the result does not depend on the number of players N .

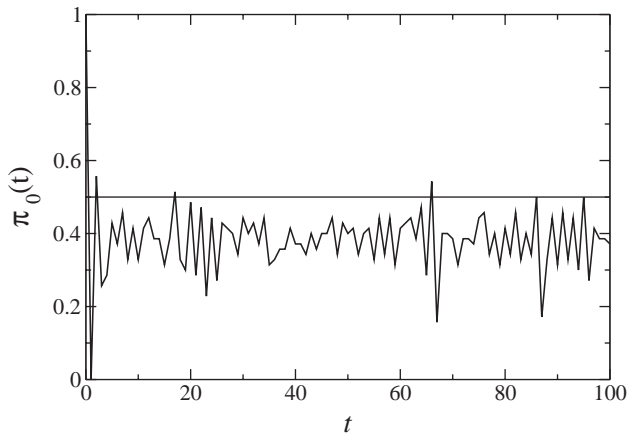


Fig. 6. Simulation showing the evolution of π_0 for $N = 70$. Fluctuations that drive π_0 above 0.5 are very rare, i.e., game A is seldom chosen.

for N around or below 20, there is an alternation of the games that can even beat the optimal periodic strategy.

We see that the MR strategy can take profit of fluctuations much better than blind strategies, but it loses all its efficiency when these fluctuations are small. We believe

that this is closely related to the second law of thermodynamics. The law prohibits any decrease of entropy only in the thermodynamic limit or for average values. On the other hand, when fluctuations are present, entropy can indeed decrease momentarily and this decrease can be exploited by a Maxwell demon.

5. History dependent games

A similar phenomenon is exhibited by the games introduced in Ref. [8], whose rules depend on the history rather than on the capital of each player. Game A is still the same as above, whereas game B is played with three coins according to the following Table 1.

Introducing a large number of players but allowing just a randomly selected fraction γ of them to vote and play, the same “voting paradox” is recovered for sufficiently small γ . Again, blind strategies achieve a constant growth of the average capital with the number of turns while the MR strategy returns a decreasing average capital, as is shown in Fig. 7. The explanation of the phenomenon goes quite along

Table 1

Before last $t - 2$	Last $t - 1$	Prob. of win at t	Prob. of loss at t
Loss	Loss	p_1	$1 - p_1$
Loss	Win	p_2	$1 - p_2$
Win	Loss	p_2	$1 - p_2$
Win	Win	p_3	$1 - p_3$

$p_1 = \frac{9}{10} - \varepsilon, p_2 = \frac{1}{4} - \varepsilon, \text{ and } p_3 = \frac{7}{10} - \varepsilon.$

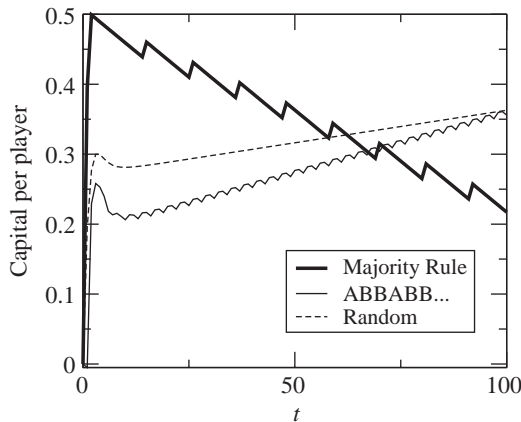


Fig. 7. Evolution of the average money of the players in the history-dependent games for $\gamma = 0.5, \varepsilon = 0.005$ and three different strategies.

the same lines as for the original games. The selfish voting prevents the necessary alternation of games, although now is game A which is selected too many times in a row.

6. Conclusions

We have shown that the paradoxical games based on the flashing ratchet exhibit a counterintuitive phenomenon when a large number of players are considered. A majority rule based on selfish voting turns to be very inefficient for large ensembles of players. We have also discussed how the rule only works for a small number of players, since in that case it is able to exploit capital fluctuations.

The inefficiency of voting encountered in our model could be rather general. The mechanism behind this inefficiency is the following. Those individuals with a capital non-multiple of three votes for game B, ensuring their own benefit in the present turn. However, even if these players conform the majority, their selfish vote is harmful for the whole ensemble. They should sacrifice their short-term profits to help the others to avoid the use of the bad coin of game B (coin 3). The blind strategy (either random or periodic) forces this sacrifice, yielding a steady gain for the ensemble. The same mechanism operates in the capital-independent games, as shown in Section 5. A similar behaviour is found in the so-called Braess paradox, where the maximization of individual profit yields overall losses for the community [18,19].

As in the Braess paradox, the model presented here shows that cooperation among individuals via a collective decision can be beneficial for everybody. In this sense, the model is also related to that presented by Toral in Ref. [17]. Since John Maynard Smith first applied game theory to biological problems [20], games have been used in ecology and social sciences as models to explain social behaviour of individuals inside a group. Some generalizations of the voting model might be useful for this purpose. For instance, it could be interesting to analyse the effect of mixing selfish and cooperative players or the introduction of players who could change their behaviour depending on the fraction of selfish voters in previous turns.

The model can also be relevant in random decision theory or the theory of stochastic control [21] since it shows how periodic or random strategies can be better than some kind of optimisation. In this sense, there has been some work on general adaptive strategies in games related with Parrondo's paradox [9,22,23].

Finally, this model and, in particular, the analysis for N finite, prompts the problem of how information can be used to improve the performance of a system. In the models presented here, information about the fluctuations of the capital is useful only for a small number of players, that is, when these fluctuations are significant. It will be interesting to analyse this crossover in further detail, not only in the case of the games but also for Brownian ratchets. Work in this direction is in progress.

Acknowledgements

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Appendix A. Evolution equations

In this section, we describe the semi-analytical solution of the model for an infinite number of players, used to depict Fig. 2. Let $\pi_i(t)$, be the fraction of players whose capital at turn t is of the form $3n + i$ with $i = 0, 1, 2$ and n an integer number.

If game A is played in turn t , these fractions change following the expression [8]:

$$\begin{pmatrix} \pi_0(t+1) \\ \pi_1(t+1) \\ \pi_2(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 1/2 + \varepsilon & 1/2 - \varepsilon \\ 1/2 - \varepsilon & 0 & 1/2 + \varepsilon \\ 1/2 + \varepsilon & 1/2 - \varepsilon & 0 \end{pmatrix} \begin{pmatrix} \pi_0(t) \\ \pi_1(t) \\ \pi_2(t) \end{pmatrix} \tag{A.1}$$

which can be written in a vector notation as

$$\vec{\pi}(t+1) = \Pi_A \vec{\pi}(t). \tag{A.2}$$

Similarly, when B is played, the evolution is given by

$$\vec{\pi}(t+1) = \Pi_B \vec{\pi}(t) \tag{A.3}$$

with

$$\Pi_B = \begin{pmatrix} 0 & 1/4 + \varepsilon & 3/4 - \varepsilon \\ 1/10 - \varepsilon & 0 & 1/4 + \varepsilon \\ 9/10 + \varepsilon & 3/4 - \varepsilon & 0 \end{pmatrix}. \tag{A.4}$$

Now we can write the evolution equation for each strategy. For the *random strategy*

$$\vec{\pi}(t+1) = \frac{1}{2}[\Pi_A + \Pi_B] \vec{\pi}(t). \tag{A.5}$$

For the *periodic strategy* (ABBABB..)

$$\vec{\pi}(3(t+1)) = \Pi_B^2 \Pi_A \vec{\pi}(3t). \tag{A.6}$$

Finally, with the MR strategy the ensemble plays game A if $\pi_0(t) \geq 1/2$ and B otherwise. Therefore,

$$\vec{\pi}(t+1) = \begin{cases} \Pi_A \vec{\pi}(t) & \text{if } \pi_0(t) \geq 1/2, \\ \Pi_B \vec{\pi}(t) & \text{if } \pi_0(t) < 1/2. \end{cases} \tag{A.7}$$

Notice that the MR strategy is the only one inducing a nonlinear evolution in the population fractions. To calculate the evolution of the capital, we compute the

winning probability in each game

$$\begin{aligned}
 p_{\text{win}}^A(t) &= \frac{1}{2} - \varepsilon \\
 p_{\text{win}}^B(t) &= \frac{1}{10}\pi_0(t) + \frac{3}{4}(1 - \pi_0(t)) - \varepsilon.
 \end{aligned}
 \tag{A.8}$$

Finally, the average capital $\langle X(t) \rangle$ per player evolves as

$$\langle X(t+1) \rangle = \langle X(t) \rangle + 2p_{\text{win}}(t) - 1
 \tag{A.9}$$

and $p_{\text{win}}(t)$ is replaced by $p_{\text{win}}^A(t)$ or $p_{\text{win}}^B(t)$, depending on the game played at turn t in each strategy.

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