Brownian motion and gambling:  
from ratchets to paradoxical games  

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Abstract  

Two losing gambling games, when alternated in a periodic or random  
fashion, can produce a winning game. This paradox has been inspired by cer-  
tain physical systems capable of rectifying fluctuations: the so-called Brow-  
nian ratchets. In this paper we review this paradox, from Brownian ratchets  
to the most recent studies on collective games, providing some intuitive ex-  
planations of the unexpected phenomena that we will find along the way.  

1 Introduction: A noisy revolution  

Imagine two simple gambling games, say A and B, in which I play against you.  
Each one is a losing game for me, in the sense that my average capital is a de-  
creasing function of the number of turns we play. Once you are convinced that I  
lose in both games, I give you a third proposal: alternate the games following the  
sequence AABBAABB... If you frown, the proposal can be modified to make it  
less suspicious: in each run we will randomly chose the game that is played. If you  
accept either of these proposals you would have trusted your intuition too much,  
not realising that random systems may behave in an unexpected way.  

The phenomenon we have just described is known as Parrondo’s paradox [1, 2,  
3]. It was originally inspired by a class of physical systems: the Brownian ratchets  
[4, 5, 6, 7, 8] and lately has received the attention of scientists working on several  
fields, ranging from biology to economics. These are systems capable of rectifying  
thermal fluctuations, such as those exhibited by a Brownian particle.
Brownian motion was one of the first crucial proofs of the discreteness of matter. First observed by Jan Ingenhousz in 1785, and later rediscovered by Brown in 1828, the phenomenon consists on the erratic or fluctuating motion of a small particle when it is embedded in a fluid. In the beginning of the XXth century, Einstein realized that these fluctuations were a manifestation of the molecular nature of the fluid\textsuperscript{1} and devised a method to measure Avogadro’s number by using Brownian motion [9]. Since then, the study of fluctuations has been a major topic in statistical mechanics.

The theory of fluctuations helped to understand noise in electrical circuits, activation processes in chemistry, the statistical nature of the second law of thermodynamics, and the origin of critical phenomena and spontaneous symmetry breaking, to cite only a few examples. In most of these cases, the role played by thermal fluctuations or thermal noise is either to trigger some process or to act as a disturbance. However, in the past two decades, the study of fluctuations has led to models and phenomena where the effect of noise is more complex and sometimes unexpected and even counterintuitive.

Noise can enhance the response of a nonlinear system to an external signal, a phenomenon known as \textit{stochastic resonance} [10]. It can create spatial patterns and ordered states in spatially extended systems [11, 12], and Brownian ratchets show that noise can be rectified and used to induce a systematic motion in a Brownian particle [4, 5, 6, 7, 8]. In these new phenomena, noise has a very different role from that considered in the past: it contributes to the creation of order. This could be relevant in several fields, and specially in biology, since most biological systems manage to keep themselves in ordered states even while surrounded by noise, both thermal noise at the level of the cell and environmental fluctuations at the macroscopic level.

However, fluctuations are not only restricted to physics, chemistry or biology. The origin of the theory of probability is closely related to gambling games, social statistics, and even to the efficiency of juries [13]. Statistical mechanics and probability theory have both contributed to each other and also to fields like economics. In 1900, five years before Einstein’s theory of the Brownian motion, the French mathematician Louis Bachelier worked out a theory for the price of a stock very similar to Einstein’s [14]. Recently this link between probability, statistical mechanics, and economics has crystallised in a new field: \textit{econophysics} [15].

Some of the aforementioned constructive role of noise has been observed in complex systems beyond physics. Stochastic resonance, for instance, has an increasing relevance in the study of perception and other cognitive processes [10, 16].

\textsuperscript{1}The thermal origin of Brownian motion was firstly proposed by Delsaux in 1877 and later on by Gouy in 1888 (see [9]).
Similarly, we expect that other elementary stochastic phenomena such as rectifica-
tion will be observed in many situations not restricted to physics.

With this idea in mind, Parrondo’s paradox came up as a translation to simple
gambling games of a Brownian ratchet discovered by Ajdari and Prost [4]. The
ratchet was afterwards named by Astumian and Bier the flashing ratchet [6] and
it was related to the idea proposed by Magnasco [5] that biological systems could
rectify fluctuations to perform work and systematic motion.

The paradox does not make use of Brownian particles, but only of the simpler
fluctuations arising in a gambling game. However, it illustrates the mechanism of
rectification in a very sharp way, and for this reason we think that it could contribute
to extend the “noisy revolution”, i.e., the idea that noise can create order, to those
fields where stochastic dynamics is relevant.

The paper is organised as follows. In section 2 we briefly review the flashing ratchet and explain how it can rectify fluctuations. Section 3 is devoted to the
original Parrondo’s paradox. There we introduce the paradoxical games as a dis-
cretisation of the flashing ratchet, discuss an intuitive explanation of the paradox
that we have called reorganisation of trends, and present an extension of the orig-
inal paradox inspired by this idea. In section 4 we introduce several versions of
the games involving a large number of players. Some interesting effects can be
observed in these collective games: redistribution of capital brings wealth [17],
and collective decisions taken by voting or by optimizing the returns in the next
turn can lead to worse performance than purely random choices [18, 19]. Finally,
in section 5 we briefly review the literature on the paradox and present our main
conclusions.

2 Ratchets

Here we revisit the flashing ratchet [4, 6], one of the simplest Brownian ratchets
and the most closely related to the paradoxical games. We refer to the exhaustive
review by Reimann on Brownian ratchets [7] or the special issue in Applied Physics
A, edited by Linke [8], for further information on the subject.

Consider an ensemble of independent one-dimensional Brownian particles in
the asymmetric sawtooth potential depicted in Fig. 1. It is not difficult to show that,
if the potential is switched on and off periodically, the particles exhibit an average
motion to the right. Let us assume that the temperature $T$ is low enough to ensure
that $kT$ is much smaller than the maxima of the potential, and that we start with
the potential switched on and with all the particles around one of its minima, as
shown in the upper plot of Fig. 1. When the potential is switched off, the particles
diffuse freely, and the density of particles spreads as depicted in the central plot of

3
the figure. If the potential is then switched on again, each particle will move back to the initial minimum or to one of the nearest neighboring minima, depending on its position. Particles within the dark region will move to the right hand minimum, those within the small grey region will move to the left hand minimum, and those within the white region will move back to their initial positions. As is apparent from the figure and due to the asymmetry of the potential, more particles fall into the right hand minimum, and thus there is a net motion of particles to the right. For this to occur, the switching can be either random or periodic, but the average period must be of the order of the time to reach the nearest barrier by free diffusion (see [4, 6] for details).

This motion can be seen as a rectification of the thermal noise associated with free diffusion. The diffusion is symmetric: some particles move to the right and some to the left, but their average position does not change. However, when the potential is switched on again, most of the particles that moved to the left are driven back to the starting position, whereas many particles that moved to the right are pushed to the right hand minimum. The asymmetric potential acts as a rectifier: it “kills” most of the negative fluctuations and “promotes” most of the positive ones.

The effect remains if we add a small force toward the left, i.e., in a direction opposite to the induced motion. In this case, the ratchet still induces a motion against the force. Consequently, particles perform work, and the system can be considered a Brownian motor. It can be proved that this type of motor is compatible with the second law of thermodynamics. In fact, the efficiency of such a motor is far below the limits imposed by the second law [20, 21]. However, the ratchet with a force exhibits a curious property: when the potential is permanently on or off, the Brownian particles move in the same direction as the force, whereas they move in the opposite direction when the potential is switched on and off. This is the essence of the paradoxical games: we have two dynamics; in each one a quantity, namely the position of the Brownian particle, decreases in average; however, the same quantity increases in average when the two dynamics are alternated.

3 Games

The flashing ratchet can be discretised in time and space, keeping most of its interesting features. The discretised version adopts the form of a pair of simple gambling games, which are the basis of the Parrondo’s paradox.
Figure 1: The flashing ratchet at work. The figure represents three snapshots of the potential and the density of particles. Initially (upper figure), the potential is on and all the particles are located around one of the minima of the potential. Then the potential is switched off and the particles diffuse freely, as shown in the centered figure, which is a snapshot of the system immediately before the potential is switched on again. Once the potential is connected again, the particles in the darker region move to the right hand minimum whereas those within the small grey region move to the left. Due to the asymmetry of the potential, the ensemble of particles move, in average, to the right.
Figure 2: Rules of the paradoxical games. In game A, the player wins (her capital increases by one euro) with a probability $1/2 - \epsilon$ and loses (her capital decreases by one euro) with a probability $1/2 + \epsilon$, $\epsilon$ being a small positive number. In the figure, these probabilities are represented by a coin with two possible outcomes. In game B, the probability to win and lose depends on the capital $X(t)$ of the player: if $X(t)$ is a multiple of three, then we use a “bad” coin, with a probability to win equal to $1/10 - \epsilon$; if $X(t)$ is not a multiple of three, then a “good” coin, with a probability to win equal to $3/4 - \epsilon$, is used. In the figure the darkness of the coins represents their “badness” for the player.

3.1 The original paradox

We consider two games, A and B, in which a player can make a bet of 1 euro. $X(t)$ denotes the capital of the player, where $t = 0, 1, 2 \ldots$ stands for the number of turns played. Game A consists of tossing a slightly biased coin so that the player has a probability $p_A$ of winning which is less than a half. That is, $p_A = 1/2 - \epsilon$, where the bias $\epsilon$ is a small positive number.

The second game, B, is played with two biased coins, a “bad coin” and a “good coin”. The player must toss the bad coin if her capital $X(t)$ is a multiple of 3, the probability of winning being $p_{bad} = 1/10 - \epsilon$. Otherwise, the good coin is tossed and the probability of winning is $p_{good} = 3/4 - \epsilon$. The rules of games A and B are represented in Fig. 2, in which the darkness represents the “badness” of each coin.

For these choices of $p_A, p_{good}$ and $p_{bad}$, both games are fair if $\epsilon = 0$, in the sense that $\langle X(t) \rangle$ is constant. This is evident for game A, since the probabilities to win and lose are equal. The analysis of game B is more involved, but we will soon prove that the effect of the good and the bad coins cancel each other for $\epsilon = 0$.

On the other hand, both games have a tendency to lose if $\epsilon > 0$, i.e., $\langle X(t) \rangle$
decreases with the number of turns $t$. Surprisingly enough, if the player randomly chooses the game to play in each turn, or plays them following some predefined periodic sequence such as ABBABB..., then her average capital $\langle X(t) \rangle$ is an increasing function of $t$, as can be seen in Fig. 3.

The paradox is closely related to the flashing ratchet. If we visualise the capital $X(t)$ as the position of a Brownian particle in a one dimensional lattice, game A, for $\epsilon = 0$, is a discretisation of the free diffusion, whereas game B resembles the motion of the particle under the action of the asymmetric sawtooth potential. Fig. 4 shows this spatial representation for game B compared with the ratchet potential. When the particle is on a dark site, the bad coin is used and the probability to win is very low, whereas on the white sites the most likely move is to the right. The sawtooth potential has a short spatial interval in which the force is negative and a long interval with a positive force. Equivalently, game B uses a bad coin on a “short interval”, i.e., on one site of every three on the lattice, and a good coin on a “long interval” corresponding to two consecutive sites which are not multiple of three (see Fig. 4).

As in the flashing ratchet, game B rectifies the fluctuations of game A. Suppose that we play the sequence AABBAABB... and that $X(t)$ is a multiple of three immediately after two instances of game B. Then we play game A twice, which can drive the capital back to $X(t)$ or to $X(t) \pm 2$. In the latter case, the next turn is for game B with a capital that is not a multiple of three, which means a good chance of winning. That is, game B rectifies the fluctuations that occurred in the
Figure 4: A random walk picture of game B compared with the ratchet potential. The bad coin (black dots) plays the role of the negative force acting on a short interval, whereas the two consecutive good coins (white dots) are the analogous of the positive force acting on the long intervals.

two turns of game A. The rectification is not as neat as in the low temperature flashing ratchet, but enough to cause the paradox.

There is a more rigorous way of associating a potential to a gambling game by using a master equation \[22\]. However, it provides a similar picture of game B, as a random walk that is nonsymmetric under inversion of the spatial coordinate and capable of rectifying fluctuations.

3.2 Reorganisation of trends

Beside the ratchet effect, one can explain the paradox considering another interesting mechanism. Recall that game B is played with two coins: a good one, used whenever the capital of the player is not a multiple of three, and a bad one which is used when the capital is a multiple of three. Therefore, the “profitability” of game B crucially depends on how often the bad coin is used, i.e., on the probability \(\pi_0\) that the capital is a multiple of three. It turns out that, when game B is played, this probability is not \(1/3\), as one could naively expect, but larger. This can be reasoned from figure 4. When the capital is at a white site, its most likely move is to the right, whereas at dark sites the most likely move is to the left. The capital thus spends more time jumping forth and back between a multiple of three and its left hand neighbour than what would do if it moved completely at random. Consequently, the probability \(\pi_0\) is larger than \(1/3\). On the other hand, under game A the capital does move in a random way. Therefore, playing game A in some turns shifts \(\pi_0\) towards \(1/3\), or, equivalently, reduces the number of times the bad coin of game B is used. In other words, game A, although losing, boosts the effect of the good coin in B, giving the overall game a winning tendency. We have named this mechanism reorganization of trends, since game A reinforces the positive trend already present in game B.

Along this section, we formulate this argument in a quantitative way. Let us
first consider game B separately. The probability to win in the $t$-th turn can be calculated as

$$p_{\text{win}}(t) = \pi_0(t)p_{\text{bad}} + [1 - \pi_0(t)]p_{\text{good}}$$  \hspace{1cm} (1)

where $\pi_0(t)$ is the probability of $X(t)$ being a multiple of 3 (i.e. of using the bad coin).

One can calculate the value of $\pi_0(t)$, by using very simple techniques from the theory of Markov chains. First, we define the random process

$$Y(t) \equiv X(t) \mod 3$$  \hspace{1cm} (2)

taking on only three possible values or states, $Y(t) = 0, 1, 2$, depending on whether the capital $X(t)$ is a multiple of three, a multiple of three plus one, or a multiple of three plus two, respectively. This variable $Y(t)$ is a Markov process, i.e., the statistical properties of $Y(t+1)$ depend only on the value taken on by $Y(t)$. This allows one to derive a master equation for its probability distribution.

Let $\pi_0(t), \pi_1(t), \pi_2(t)$ be the probability that $Y(t)$ is equal to 0, 1, and 2, respectively. There are two possibilities for $Y(t) = 2$ to occur: either $Y(t-1) = 0$ and we lose in the $t$-th turn (with probability $1 - p_{\text{bad}}$), or $Y(t-1) = 1$ and we win in the $t$-th turn (with probability $p_{\text{good}}$). Therefore:

$$\pi_2(t) = (1 - p_{\text{bad}})\pi_0(t-1) + p_{\text{good}}\pi_1(t-1).$$  \hspace{1cm} (3)

Following the same type of argument, one can derive equations for $\pi_0(t)$ and $\pi_1(t)$, and the three equations can be written in matrix form as:

$$\vec{\pi}(t) = \Pi_B \vec{\pi}(t - 1)$$  \hspace{1cm} (4)

where

$$\vec{\pi}(t) \equiv \begin{pmatrix} \pi_0(t) \\ \pi_1(t) \\ \pi_2(t) \end{pmatrix}$$  \hspace{1cm} (5)

and

$$\Pi_B \equiv \begin{pmatrix} 0 & 1 - p_{\text{good}} & p_{\text{good}} \\ p_{\text{bad}} & 0 & 1 - p_{\text{good}} \\ 1 - p_{\text{bad}} & p_{\text{good}} & 0 \end{pmatrix}.$$  \hspace{1cm} (6)

After a small number of turns of game B, $\vec{\pi}(t)$ approaches to a stationary value $\vec{\pi}_{\text{st}}$, which is invariant under the transformation given by Eq. (4), i.e.:

$$\vec{\pi}_{\text{st}} = \Pi_B \vec{\pi}_{\text{st}}.$$  \hspace{1cm} (7)
The third component of the solution of this equation reads:

$$\pi_{0B}^{st} = \frac{5}{13} - \frac{440}{2197} \epsilon + O(\epsilon^2) \simeq 0.38 - 0.20 \epsilon$$  \hspace{1cm} (8)$$

where we have used the values of the original paradox, $p_{\text{bad}} = \frac{1}{10} - \epsilon$ and $p_{\text{good}} = \frac{3}{4} - \epsilon$, and have expanded the solution up to first order of $\epsilon$, to simplify the exposition.

Substituting this value in Eq. (1) we obtain the probability of winning for game B for sufficiently large $t$

$$p_{\text{win},B} = \frac{1}{2} - \frac{147}{169} \epsilon + O(\epsilon^2)$$  \hspace{1cm} (9)$$

which is less than $1/2$ for $\epsilon > 0$. This proves that game B is fair for $\epsilon = 0$ and losing for $\epsilon > 0$, as shown in Fig. 3.

The paradox arises when game A comes into play. Game A is always played with the same coin, regardless of the value of the capital $X(t)$, and therefore drives the probability distribution $\bar{\pi}(t)$ to a uniform distribution. Thus, game A makes $\pi_0(t)$ tend to $1/3$. Since $1/3 < \pi_{0B}^{st}$, the effect of game A is to decrease the probability of using the bad coin in the turns where B is played.

This can be seen in a more precise way, since the random combination of games A and B can be again solved by using the master equation:

$$\bar{\pi}_{AB}^{st} = \frac{1}{2} [\Pi_B + \Pi_A] \pi_{AB}^{st}$$  \hspace{1cm} (10)$$

where

$$\Pi_A = \begin{pmatrix} 0 & 1 - p_A & p_A \\ p_A & 0 & 1 - p_A \\ 1 - p_A & p_A & 0 \end{pmatrix}$$ \hspace{1cm} (11)$$

with $p_A = 1/2 - \epsilon$. The probability of using the bad coin decreases to

$$\pi_{0AB}^{st} = \frac{245}{709} - \frac{48880}{502681} \epsilon + O(\epsilon^2) \simeq 0.35 - 0.10 \epsilon.$$  \hspace{1cm} (12)$$

The probability of winning in this randomised combination of games A and B is

$$p_{\text{win},AB} = \frac{\pi_{0AB}^{st}}{2} \frac{p_{\text{bad}} + p_A}{2} + [1 - \pi_{0AB}^{st}] \frac{p_{\text{good}} + p_A}{2}$$

$$= \frac{727}{1418} - \frac{486795}{502681} \epsilon + O(\epsilon^2)$$ \hspace{1cm} (13)$$

which is greater than $1/2$ for a sufficiently small $\epsilon$. 

10
This is the mechanism behind the paradox which we have termed “reorganisation of trends”: although game A consists itself in a negative trend because it uses a slightly bad coin, it increases the probability of using the good coin of B, i.e., game A reinforces the positive trend already present in B enough to make the combination win.

Periodic sequences can also be studied as Markov chains and their probability of winning in a whole period can be easily computed using different combinations of the matrices $\Pi_A$ and $\Pi_B$. Finally, the slopes of the curves in Fig. 3 can be calculated as $\langle X(t + 1) \rangle - \langle X(t) \rangle = 2p_{\text{win}} - 1$.

### 3.3 Capital-independent games

The modulo rule in game B is quite natural in the original representation of the games as a Brownian ratchet. However, the rule may not suit some applications of the paradox to biology, biophysics, population genetics, evolution, and economics. Thus, it would be desirable to devise new paradoxical games based on rules independent of the capital. Parrondo, Harmer and Abbott introduced such a game in Ref. [23], inspired by the reorganisation of trends explained in the last section.

In the new version, game A remains the same as before, but a game $B'$, which depends on the history of wins and losses of the player, is introduced. Game $B'$ is played with four coins $B'_1, B'_2, B'_3, B'_4$ following history-based rules explained in table 1.

<table>
<thead>
<tr>
<th>Before last</th>
<th>Last</th>
<th>Coin</th>
<th>Prob. of win at $t$</th>
<th>Prob. of loss at $t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loss</td>
<td>Loss</td>
<td>$B'_1$</td>
<td>$p_1$</td>
<td>$1 - p_1$</td>
</tr>
<tr>
<td>Loss</td>
<td>Win</td>
<td>$B'_2$</td>
<td>$p_2$</td>
<td>$1 - p_2$</td>
</tr>
<tr>
<td>Win</td>
<td>Loss</td>
<td>$B'_3$</td>
<td>$p_3$</td>
<td>$1 - p_3$</td>
</tr>
<tr>
<td>Win</td>
<td>Win</td>
<td>$B'_4$</td>
<td>$p_4$</td>
<td>$1 - p_4$</td>
</tr>
</tbody>
</table>

Table 1: History-based rules for game B’

The paradox reappears, for instance, when setting $p_1 = 9/10 - \epsilon$, $p_2 = p_3 = 1/4 - \epsilon$, and $p_4 = 7/10 - \epsilon$. With these numbers and for $\epsilon$ small and positive, $B'$ is a losing game, while either a random or a periodic alternation of A and $B'$ produces a winning result. Fig. 5 shows a theoretical computation of the average capital for these history-dependent paradoxical games.

The paradox is reproduced because there are bad coins in game $B'$ which are played more often than in a completely random game, i.e., a quarter of the time.
Figure 5: Average capital as a function of the number of turns in the capital independent games. We plot the result for game A and B’, as well as for the random combination and the periodic sequence AAB’B’... In all the cases, $\epsilon = 0.003$.

For the above choices of $p_i$, $i = 1, 2, 3, 4$, the bad coins are $B_2'$ and $B_3'$. The other two coins, $B_1'$ and $B_4'$, are good coins.

Due to the fact that game B’ rules depend on the history of wins and losses, the capital $X(t)$ is no longer a Markovian process. However, the random vector

$$Y(t) = \begin{pmatrix} X(t) - X(t-1) \\ X(t-1) - X(t-2) \end{pmatrix}$$

(14)

can take on four different values and is indeed a Markov chain. The transition probabilities are again easily obtained from the rules of game B’ and an analytical solution can be obtained following a similar argument as in section 3.2 (see however Ref. [23] for details).

We see that the mechanism that we have called reorganisation of trends can be used to extend the paradox to other gambling games. It is also noteworthy that the price we must pay to eliminate the dependence on the capital in the original paradox is to consider history-dependent rules, i.e., games where the capital is no longer Markovian.

4 Collective games

In this section we analyse three different versions of paradoxical games played by a large number of individuals. The three share the feature that it can sometimes be
better for the players to sacrifice short term benefits for higher returns in the future.

4.1 Capital redistribution brings wealth.

Reorganisation of trends tells us that the essential role of game A in the paradox is to randomise the capital and make its distribution more uniform. Toral has found that a redistribution of the capital in an ensemble of players has the same effect [17].

In the new paradoxical games introduced by Toral in [17], there are $N$ players and one of them is randomly selected in each turn. They can play two games. The first one, game $A'$, consists of giving a unit of his capital to another randomly chosen player in the ensemble, that is, game $A'$ is nothing but a redistribution of the total capital. The second one, game $B$, is the same as in the original paradox. Under game $A'$ the capital does not change, where game $B$ is, as before, a losing game. The striking result is that the random combination of the two games is winning, i.e., the redistribution of capital performed in the turns where $A'$ is played turns the losing game $B$ into a winning one, actually increasing the total capital available. Thus, the redistribution of capital turns out to be beneficial for everybody. This effect is shown in Fig. 6 where the average total capital in a simulation with 10 players and 500 realizations is depicted for games $B$ and $A'$, and for their random combination. It is remarkable that the effect is still present when the capital is required to flow from richer to poorer players (see [17] for details).

The explanation to this phenomenon follows the same lines as in the original paradox.

4.2 Dangerous choices I: The voting paradox

Up to now, we have considered sequences of games that are “imposed” to the player or players. Either they play game A, game B, or a periodic or random sequence, but we never allow the players to choose the game to be played in each turn. In the case of a single player this deference is quite generous, since her capital would increase in average under the following trivial choice: she selects game B if her capital is not a multiple of three and game A otherwise. This is undoubtedly the best strategy, because the best coin is always used in every turn. Moreover, it is not difficult to see that this choice strategy performs much better than any other random or periodic combination of games.

However, things change when we consider an ensemble of players. How can the ensemble choose the game to be played in each turn? There are some possibilities, such as letting them vote for the preferred game or trying to maximise the winning probability in each turn. Which is then the best choice strategy? We
will see that the paradoxical games also yield some surprises in this context: the choice that prefers the majority of the ensemble turns to be worse than a random or periodic combination of games. Even if we select the game maximising the profit in every turn, we can end with systematic loses, as shown in the next section.

Consider a set of $N$ players who play game A or B against a casino. In each turn, all of them play the same game. Therefore, they have to make a collective decision, choosing between game A or B in each turn. We will firstly use a majority rule to select the game, that is, the game which receives more votes is played by all the players simultaneously. The vote of each player will be determined by her capital, following the strategy that we have explained above for a single player. Players with capital multiple of three will vote for game A, whereas the rest will vote for B.

This strategy, which is optimal for a single player, turns to be losing if the number of players is large enough [19]. This is shown in figure 7, where we plot the average capital in an ensemble with an infinite number of players. On the other hand, if the game is selected at random the capital increases in time.

In order to explain this behaviour, we will focus on the evolution of $\pi_0(t)$, the fraction of players whose capital is a multiple of three. The selection of the game by voting can be rephrased in terms of $\pi_0(t)$. As mentioned above, every player votes for the game which offers him the higher probability of winning according
to his own state. Then, every player whose capital is a multiple of three will vote for game A in order to avoid the bad coin in B. That accounts for a fraction $\pi_0(t)$ of the votes. The remaining fraction $1 - \pi_0(t)$ of the players will vote for game B to play with the good coin. Since the majority rule establishes that the game which receives more votes is selected, game A will be played if $\pi_0(t) > 1/2$. Conversely, the whole set of players will play game B when $\pi_0(t)$ is below 1/2.

On the other hand, as we have seen in section 3.2, playing game B makes $\pi_0(t)$ tend to a stationary value given by Eq. (8), namely, $\pi_{0B}^{st} \simeq 0.38 - 0.2\epsilon < 1/2$ for $\epsilon > 0$, whereas playing game A makes $\pi_0$ tend to 1/3. This is still valid for the present model, since the $N$ players only interact when they make the collective decision, otherwise they are completely independent.

If $\pi_0(t) > 1/2$, then the ensemble of players will select game A. The fraction $\pi_0(t)$ will decrease until it crosses this critical value 1/2. At that turn, B is the selected game and it will remain so as long as $\pi_0$ does not exceed 1/2. However, this can never happen, since game B drives $\pi_0$ closer and closer to $\pi_{0B}^{st}$, which is below 1/2. Hence, after a number of turns, the system gets trapped playing game B forever with $\pi_0$ asymptotically approaching $\pi_{0B}^{st}$. Since $\epsilon$ is positive, game B is a losing game (c.f. section 3.2) and, therefore, the majority rule yields systematic losses, as can be seen in Fig. 7. We have also plotted in Fig. 8 the fraction $\pi_0(t)$, to check that, once $\pi_0(t)$ crosses 1/2, game B is always chosen and $\pi_0(t)$ approaches $\pi_{0B}^{st}$, staying far below 1/2.

On the other hand, if, instead of using the majority rule, we select the game at random or following a periodic sequence, game A will be chosen even though $\pi_0 < 1/2$. This is a bad choice for the majority of the players, since playing B would make them toss the good coin. That is, the random or periodic selection will contradict from time to time the will of the majority. Nevertheless, choosing the game at random keeps $\pi_0$ away from $\pi_{0B}^{st}$, as shown in Fig. 8, i.e., in a region where game B is winning ($\pi_0 < \pi_{0B}^{st}$). Therefore, the random choice yields systematic gains, as shown in Fig. 7.

It is worth noting that choosing the game at random is exactly the same as if every player voted at random. Therefore, the players get a winning tendency when they vote at random whereas they lose their capital when they vote according to their own benefit in each run.

### 4.3 Dangerous choices II: The risks of short-range optimisation

Yet another “losing now to win later” effect can be observed in the collective paradoxical games with another choice strategy. As in the previous example, we consider a large set of players, but we have to add a small ingredient to achieve the desired effect: now only a randomly selected fraction $\gamma$ of them play the game in
each turn. Suppose we know the capital of every player so we can compute which game, A or B, will give the larger average payoff in the next turn. Again, and even more strikingly, selecting the “most favorable game” results in systematic losses whereas choosing the game at random or following a periodic sequence steadily increases the average capital [18].

The knowledge of the capital of every player allows us to choose the game with the highest average payoff in the next turn, since this optimal game can easily be obtained from the fraction \( \pi_0(t) \) of players whose capital is a multiple of three. These individuals will play the bad coin if game B is chosen and the remaining fraction \( 1 - \pi_0(t) \) will play the good coin. Hence, the probability of winning for game B reads

\[
 p_{\text{winB}} = \pi_0 p_{\text{bad}} + (1 - \pi_0) p_{\text{good}}. \tag{15}
\]

In case game A is selected, the probability to win is

\[
 p_{\text{winA}} = p_A = 1/2 - \epsilon \quad \text{for all time } t. 
\]

Therefore, to choose the game with the larger payoff \( \langle X(t+1) \rangle - \langle X(t) \rangle = 2p_{\text{win}} - 1 \) in every turn \( t \), we must

\[
\begin{align*}
\text{play A} & \quad \text{if } \quad p_{\text{winA}} \geq p_{\text{winB}}(\pi_0) \\
\text{play B} & \quad \text{if } \quad p_{\text{winA}} < p_{\text{winB}}(\pi_0) \tag{16}
\end{align*}
\]

Figure 7: Average capital per player in the collective games as a function of the number of turns, when the game is selected at random and following the preference of the majority of the players (MR). Notice that, in the stationary regime, the majority rule (MR) yields systematic loses whereas the random choice wins in average. These are analytical results with \( \epsilon = 0.005 \) and an infinite number of players.
Figure 8: The fraction of players $\pi_0(t)$ with capital multiple of three as a function of time when the game is chosen at random and following the majority rule (MR). In both cases, $\epsilon = 0.005$ and $N = \infty$. The horizontal lines indicate the threshold value for the majority rule (1/2), and the stationary values for games A and B, $\pi_{0A}^{\text{st}}$ and $\pi_{0B}^{\text{st}}$, respectively. The figure clearly shows that the random strategy keeps $\pi_0(t)$ small, whereas the majority rule, selecting B most of the time, drives $\pi_0(t)$ to a value where both game A and B are losing.
Figure 9: Average capital as a function of time for the three different strategies explained in the text, with $N = \infty$, $\gamma = 0.5$, and $\epsilon = 0.005$. The short-range (SR) optimal strategy is losing in the stationary regime, whereas the two blind strategies: choosing the game to be played either at random or following the periodic sequence (ABBABB...), yield a systematic gain.

or equivalently

\[
\begin{align*}
\text{play A} & \quad \text{if} \quad \pi_0(t) \geq \pi_{0c} \\
\text{play B} & \quad \text{if} \quad \pi_0(t) < \pi_{0c}
\end{align*}
\]

with $\pi_{0c} \equiv (p_A - p_{\text{good}})/(p_A - p_{\text{bad}}) = 5/13$. We will call this way of selecting the game the short-range optimal strategy. We will also consider that the game is selected following either a random or periodic sequence. These are both blind strategies, since they do not make any use of the information about the state of the system. However, and surprisingly enough, they turn out to be much better than the short-range optimal strategy, as shown in Fig. 9.

Notice that (17) is similar to the way the game is selected by the majority rule considered in the previous section, but replacing $1/2$ by the new critical value $\pi_{0c} = 5/13$. Therefore, the explanation of this model goes quite along the same lines as for the voting paradox, although with some differences. Unlike $1/2$, $\pi_{0c}$ equals the stationary value of $\pi_0(t)$ for game B when $\epsilon = 0$. As in the voting paradox, game A drives $\pi_0$ below $\pi_{0c}$ because game A makes $\pi_0$ tend to $1/3$. If $\pi_0(t) < \pi_{0c}$, then game B is played, but $\pi_0(t + 1)$ will be still below $\pi_{0c}$ only for
Figure 10: The fraction $\pi_0(t)$ of players with capital multiple of three as a function of the number of turns, for $\epsilon = 0$, $N = \infty$, and $\gamma = 0.5$. The horizontal lines show the stationary values for game A and game B (which coincides with the critical fraction $\pi_{0c}$ for the short-range optimal strategy). As we have in figure 8 with the majority rule, the short-range optimal strategy drives $\pi_0(t)$ towards higher values than the other two strategies.

$\gamma$ sufficiently small. For example, if $\gamma = 1/2$ and $\epsilon = 0$, game B is chosen forty times in a row before switching back to game A, making $\pi_0$ become approximately equal to $\pi_{0B}^{st}$ at almost every turn. This behaviour is shown in Fig. 10. As long as $\pi_0$ is close to $\pi_{0B}^{st}$, the average capital remains approximately constant, as shown in Fig. 11.

In contrast, the periodic and random strategies choose game A with $\pi_0 < \pi_{0c}$. Although this does not produce earnings in that turn, it keeps $\pi_0$ away from $\pi_{0B}^{st}$. When game B is chosen again, it has a large expected payoff since $\pi_0$ is not close to $\pi_{0B}^{st}$. By keeping $\pi_0$ not too close to $\pi_{0B}^{st}$, the blind strategies perform better than the short-range optimal prescription, as can be seen in Fig. 11.

The introduction of $\epsilon > 0$ has two effects. First of all, it makes $\pi_{0B}^{st}$ decrease by a small amount, as indicated in Eq. (8). This makes it even more difficult for the short-range optimal strategy to choose game A, and after a few runs game B is always selected. Since game B is now a losing game, the short-range optimal strategy is also losing whereas periodic and random strategies keep their winning tendency, as can be seen in Fig. 9.

To summarise, the short-range optimal strategy chooses B most of the time,
Figure 11: Average capital as a function of time for the three different strategies explained in the text and for $\epsilon = 0$, $N = \infty$ and $\gamma = 0.5$. In this case the short-range optimal strategy is still winning, due to the small jumps coinciding with the selection of game A. However, most of the turns game B is played with a value of $\pi_0(t)$ very close to $\pi_{0B}^\ast$. 
since it is the game which gives the highest returns in each turn. However, this choice drives $\pi_0(t)$ to a region in which $B$ is no longer a winning game. On the other hand, the random strategy from time to time sacrifices the short term returns by selecting game $A$, but this choice keeps the system in a “productive region”. We could say that the short-range optimal strategy is “killing the goose that laid the golden eggs”, an effect that is also present in simple deterministic systems [18].

5 Conclusions

We have presented the original Parrondo’s paradox and several examples showing how the basic mechanisms underlying the paradox can yield other counter-intuitive phenomena. We finish by reviewing these mechanisms as well as the literature related with the paradox.

The first mechanism, the ratchet effect, occurs when fluctuations can help to surmount a potential barrier or a “losing streak”. These fluctuations can either come from another losing game, such as in the original paradox, from a redistribution of the capital, such as in Toral’s collective games [17], or from a purely diffusive motion, such as in the flashing ratchet.

A second mechanism is the reorganisation of trends, which occurs when game $A$ reinforces a positive trend already present in game $B$. The same mechanism can be observed in the games with capital independent rules and it helps to understand the counter-intuitive behaviour of the collective games presented in section 4.2 and 4.3, where random choices perform better than the choice preferred by the majority or the one optimizing short term returns. These models also prompt the question of how information can be used to design a strategy. It is a relevant question for control theory and also for statistical mechanics, since the paradox is a purely collective effect that goes away for a single player, i.e., the choices following the short-range optimal strategy and the majority rule perform much better than the random or periodic ones.

There is a third mechanism which we have not addressed along the paper, but immediately arises if we consider the games as dynamical systems: the outcome of an alternation of dynamics can always be interpreted as a stabilization of transient states. This interpretation has allowed some authors to extend the basic message of the paradox to pattern formation in spatially extended systems [24, 25, 26, 27]. In these papers, a new mechanism of pattern formation based on the alternation of dynamics is introduced. They show how the global alternation of two dynamics, each of which leads to a homogeneous steady state, can produce stationary or oscillatory patterns upon alternation.

Another interesting application of the stabilization of transient states is pre-
presented in Ref. [28]. Two dynamics for the population of a virus are introduced with the following property: in each dynamics, the population vanishes, whereas the alternation of the two dynamics, whose origin could be the seasonal variation, induces an outbreak of the virus.

Similar effects can be seen in quantum systems. Lee et al. have devised a toy model in which the alternation of two decoherence dynamics can significantly decrease the decoherence rate of each separate dynamics [29]. Also in the quantum domain, the paradox has received some attention: there have been some proposals of a quantum version of the games [30, 31] closely related with the recent theory of quantum games [32], and the paradox has been reproduced in the contexts of quantum lattice gases [33] and quantum algorithms [34].

To finish this partial account of the existing literature on the paradox, we mention the work by Arena et al [35], who analyse the performance of the games using chaotic instead of random sequences of choices: that of Chang and Tsong [36], who study the hidden coupling between the two games in the paradox and present several extensions even for deterministic dynamics; and the paper by Kocarev and Tasev [37], relating the paradox with Lyapunov exponents and stochastic synchronisation.

In summary, Parrondo’s paradox has drawn the attention of many researchers to non-trivial phenomena associated with switching between two dynamics. We have tried to reveal in this paper some of the basic mechanisms that can yield an unexpected behaviour when switching between two dynamics, and how these mechanisms work in several versions of the paradox. As mentioned in the introduction, we believe that the paradox and its extensions are contributing to a deeper understanding of stochastic dynamical systems. In the case of statistical mechanics, switching is in fact a source of non-equilibrium which is ubiquitous in nature, due to day-night or seasonal variations [28]. Nevertheless, it has not been studied in depth until the recent introduction of ratchets and paradoxical games. As the paradox suggests, we will probably see in the future new models and applications confirming that noise and switching, even between equilibrium dynamics, can be a powerful combination to create order and complexity.

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References


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