New Paradoxical Games Based on Brownian Ratchets

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Based on Brownian ratchets, a counterintuitive phenomenon has recently emerged—namely, that two losing games can yield, when combined, a paradoxical tendency to win. A restriction of this phenomenon is that the rules depend on the current capital of the player. Here we present new games where all the rules depend only on the history of the game and not on the capital. This new history-dependent structure significantly increases the parameter space for which the effect operates.

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In the early 1990s it was shown that a Brownian particle in a periodic and asymmetric potential moves to the right (say) in a systematic way when the potential is switched on and off, either periodically or randomly [1,2]. This so-called *flashing ratchet* is in the class of phenomena known as Brownian ratchets [3]. The flashing ratchet can be viewed as the combination of two dynamics: Brownian motion in an asymmetric potential and Brownian motion on a flat potential. In each of these two cases, the particle does not exhibit any systematic motion. However, when they are alternated the particle moves to the right. The effect persists even if we add a uniform external force pointing to the left. In that case, the two dynamics discussed above yield motion to the left but when they are combined, the particle moves to the right.

It has recently been shown, in the seminal papers [4–7], that a discrete-time version of the flashing ratchet can be interpreted as simple gambling games. Here we have two losing games which become winning when combined. These games are the simplest situation of a paradoxical mechanism which, we believe, can be present in many situations of interest. The apparent paradox points out that if one combines two dynamics in which a given variable decreases the same variable can increase in the resulting dynamics. Examples of related phenomena include enzyme transport analyzed by a four-state rate model [8], finance models where capital grows by investing in an asset with *negative* typical growth rate [9], stability produced by combining unstable systems [10], counterintuitive drift in the physics of granular flow [11], the combination of declining branching processes producing an increase [12], and counterintuitive drift in switched diffusion processes in random media [13].

The games originally described in [4–7] are expressed in terms of tossing biased coins. The games rely on a state-dependent rule based on the player's capital and two losing games can surprisingly combine to win. This effect was shown to be essentially a discrete-time Brownian ratchet [4]. This is of interest to information theorists who have long studied the problem of producing a fair game from biased coins [14] and winning games from fair games [15], inspired by the work of von Neumann [16]—the games we are discussing go a step further, demonstrating a winning expectation produced from losing games and have recently been analyzed from the point of view of information theory [17]. Seigman [18,19] has reinterpreted the capital of the games in terms of electron occupancies in energy levels, recasting the problem in terms of rate equations. Similarly, Van den Broeck *et al.* [7] have likened the analysis of the transition probabilities of the games to Onsager's treatment of reaction rates in circular chemical reactions [20]. It has been suggested in [6] that an area of interest to quantum information theory would be to recast the games in terms of quantum probability amplitudes along the lines of [21–23]. Quantum ratchets have now been experimentally realized [24] and thus quantum game theory based on ratchets is of interest.

However, one of the limitations of the game paradox and its applicability to further situations is that it relies on a modulo rule based on the capital of the player. The modulo arithmetic rule is quite natural for an interpretation of the paradox in terms of energy levels; however, for processes in biology and biophysics it is unnatural. Applicability of the paradox to population genetics, evolution, and economics has been suggested [25] and thus a desirable version of the paradox would be to have rules independent of capital.

In this Letter we present a new interpretation of the paradox in terms of good and bad biased coins which are played more or less often when the two games are combined. This interpretation allows us to introduce an important modification to the original games, namely, games which do not depend on the capital but only on the recent history of wins and losses.

The two original games are as follows. The player has some capital $X(t)$, $t = 0, 1, 2, \ldots$. In game A the capital is increased by one with probability p and decreased by one with probability $1 - p$. In game B, the rules are

	Prob. of win	Prob. of loss
$X(t)/3 \in \mathbb{Z}$	p_1	$1 - p_1$
$X(t)/3 \notin \mathbb{Z}$	p_2	$1 - p_2$

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Here "win" means increasing the capital by one and "loss" means decreasing it by one. For the choice, $p = 1/2 - \epsilon$, $p_1 = 1/10 - \epsilon$, and $p_2 = 3/4 - \epsilon$, with $\epsilon > 0$, the two games have a tendency to lose. More precisely $\langle X(t)\rangle$ is a decreasing function of the number of runs t. However, if in each run we randomly choose the game we play, then, for ϵ small enough, $\langle X(t)\rangle$ is an increasing function of t.

An explanation of this paradox is as follows. First, let us imagine the above rules as implemented by three biased coins, A, B_1, and B_2, with probability for tails p, p_1, and p_2, respectively. We see that A and B_1 are "bad coins," whereas B_2 is a "good coin" for the player. When game B is played alone, at first sight one would say that B_1 is used one-third of the time. However, this is not the case. When the capital is a multiple of 3, $X(t) = 3n$, there is a high probability of losing; i.e., $X(t+1) = 3n - 1$ is the most likely value for the capital at $t+1$. If this is the case, we have to use coin B_2 in the $t+1$ run and the most likely outcome is now a win. Therefore, the most likely capital at $t+2$ is again $X(t+2) = 3n$. We see that the probability of $X(t)$ being a multiple of 3 is bigger than 1/3, due to the very rules of game B. The precise value of the equilibrium probability can be calculated by defining the Markov process $Y(t) \equiv X(t) \bmod 3$, which takes on only three values, $Y(t) = 0, 1, 2$. The stationary distribution for $Y(t)$, when $\epsilon = 0$ is given by $\pi_0 = \frac{5}{13}$, $\pi_1 = \frac{2}{13}$, and $\pi_2 = \frac{6}{13}$. The fairness of the game is indicated by $\pi_0 p_1 + (\pi_1 + \pi_2)p_2 = 1/2$.

When coin A comes to play, the stationary distribution changes. For instance, if games A and B are switched at random, one has $\pi_0' = \frac{245}{709}$, $\pi_1' = \frac{180}{709}$, and $\pi_2' = \frac{284}{709}$. The game is no longer fair because $\pi_0' = 245/709 = 0.346$ is closer to 1/3 than $\pi_0 = 0.385$, for the bad coin and now the good coin, B_2, is played more often than before. The effect persists even if coin A is bad, leading to the paradox.

This interpretation helps us to find a new version of the paradox with capital-independent games. Game A is the same as before and we introduce game B′ which is played with four coins: B_1', B_2', B_3', and B_4'. Which coin is used now depends on the history of the game:

Before last $t-2$	Last $t-1$	Coin at t	Prob. of win at t	Prob. of loss at t
Loss	Loss	B_1'	p_1	$1 - p_1$
Loss	Win	B_2'	p_2	$1 - p_2$
Win	Loss	B_3'	p_3	$1 - p_3$
Win	Win	B_4'	p_4	$1 - p_4$

This is in fact the most general game depending on the outcome of the two last runs. The paradox could even be reproduced with this type of game if the bad coins in game B′ are played more often than what is expected in a completely random game, i.e., one-quarter of the time.

Notice that the capital $X(t)$ in game B′ is not a Markovian process. However, one can define the vector

$$Y(t) = \begin{pmatrix} X(t) - X(t-1) \\ X(t-1) - X(t-2) \end{pmatrix}, \tag{1}$$

which can take four values $(\pm 1, \pm 1)$ and does form a Markov chain. The transition probabilities are easily obtained from the rules of game B′. Let $\pi_1(t)$, $\pi_2(t)$, $\pi_3(t)$, and $\pi_4(t)$ be the probabilities that $Y(t)$ is $(-1,-1)$, $(1,-1)$, $(-1,1)$, and $(1,1)$, respectively. The probability distribution $\vec{\pi}(t)$ verifies the evolution equation: $\vec{\pi}(t+1) = \mathbf{A}\vec{\pi}(t)$, where the matrix $\mathbf{A}$ is given by the transition probabilities and reads

$$\mathbf{A} = \begin{pmatrix} 1-p_1 & 0 & 1-p_3 & 0 \\ p_1 & 0 & p_3 & 0 \\ 0 & 1-p_2 & 0 & 1-p_4 \\ 0 & p_2 & 0 & p_4 \end{pmatrix}. \tag{2}$$

The stationary distribution $\vec{\pi}_{\text{st}}$ of this Markov chain is by definition invariant under the action of the matrix $\mathbf{A}$, i.e., $\vec{\pi}_{\text{st}}\mathbf{A} = \vec{\pi}_{\text{st}}$. This distribution reads

$$\vec{\pi}_{\text{st}} = \frac{1}{N}\begin{pmatrix} (1-p_3)(1-p_4) \\ (1-p_4)p_1 \\ (1-p_4)p_1 \\ p_1 p_2 \end{pmatrix}, \tag{3}$$

where N is a normalization constant.

In the stationary regime, the probability to win in a generic run is

$$\begin{aligned} p_{\text{win}} &= \sum_{i=1}^{4} \pi_{\text{st},i}\, p_i \\ &= \frac{p_1(p_2 + 1 - p_4)}{(1-p_4)(2p_1 + 1 - p_3) + p_1 p_2}, \end{aligned} \tag{4}$$

which can be rewritten as $p_{\text{win}} = 1/(2 + c/s)$, with $s = p_1(p_2 + 1 - p_4) > 0$ for any choice of the rules, and $c = (1-p_4)(1-p_3) - p_1 p_2$.

Therefore, the tendency of game B′ obeys the following rule: if $c < 0$, B′ is winning; if $c = 0$, B′ is fair; and if $c > 0$, B′ is losing. Again, here losing, winning, and fair mean that $\langle X(t)\rangle$ is, respectively, a decreasing, increasing, or constant function of t.

Since when game B′ is combined with game A the vector $Y(t)$ as defined in Eq. (1) is still a Markov chain, the same procedure applies. The probabilities of winning are now replaced by $p_i' = (p_i + p)/2$. Summarizing, to reproduce the paradox with capital-independent games we have to find a set of five numbers, p and p_i $(i = 1,2,3,4)$, such that

$$\begin{gathered} 1 - p > p\,, \\ (1-p_4)(1-p_3) > p_1 p_2\,, \\ (2 - p_4 - p)(2 - p_3 - p) < (p_1 + p)(p_2 + p)\,, \end{gathered} \tag{5}$$

where the third equation is just the second with p_i' and the inequality reversed (to make the combined game winning instead of losing).

One of the coins in game B′ must be bad and used more often than one-quarter of the time. It cannot be either

B_1' or B_4' because the probability of using these coins depends on whether the game is losing or winning (if B_1' is played more often than B_4', it is obvious that the game is losing). The bad coins should be B_2' and B_3'. Let us set $p = 1/2 - \epsilon$, $p_1 = 9/10 - \epsilon$, $p_2 = p_3 = 1/4 - \epsilon$, and $p_4 = 7/10 - \epsilon$. With these numbers, one can see that the two first inequalities in Eq. (5) are always satisfied if $\epsilon > 0$, whereas the third is satisfied if $\epsilon < 1/168 = 0.005\,95$—i.e., the paradox occurs when $0 < \epsilon < 1/168$, for our chosen parameter set in this example.

The simulation in Fig. 1 shows that as games A and B′ evolve individually the capital declines, as expected (i.e., they are losing games). On the same graph we see the remarkable result that when A and B′ are alternated either randomly or periodically, the capital now increases. This reproduces the paradoxical behavior first observed in the original games [4], but now without state dependence on capital. The slopes of the curves corresponding to game B′ and to the random combination can easily be calculated as $\langle X(t+1)\rangle - \langle X(t)\rangle = 2p_{\text{win}} - 1$, with p_{win} given by Eq. (4). The old and new games have a fundamental difference in that the old ones can be interpreted in terms of a random walk in a periodic environment [19] or a Brownian particle in a periodic potential, whereas the rules of the present games are homogeneous. We could say that the periodic structure of the original games has been transferred to the memory of the rules in the new games. Therefore, the paradox needs at least one of these two ingredients: inhomogeneity or non-Markovianity [26].

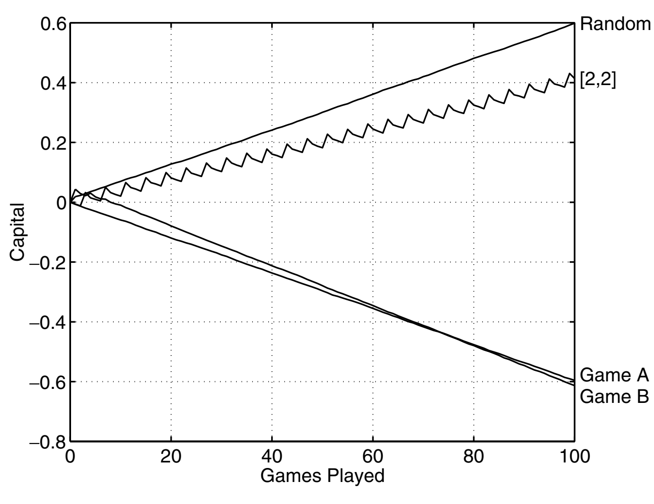

FIG. 1. Evolution of capital with play. The lower two curves show that games A and B′ lose when individually played. [2,2] indicates game A played 2 times followed by game B′ played 2 times and so on. The top curve indicates random switching between games A and B′. Capital surprisingly increases in the random or periodic cases. Simulations are carried out with $\epsilon = 0.003$, with averaging over 500 000 ensembles. $p = 1/2 - \epsilon$, $p_1 = 9/10 - \epsilon$, $p_2 = p_3 = 1/4 - \epsilon$, and $p_4 = 7/10 - \epsilon$. There are four possible initial conditions—these affect the offsets but not the slopes—all the above curves are the average of the four cases.

Consider now a periodic combination of games A and B′. Figure 2 shows the capital after 500 games—where game A is played a times and game B′ is played b times. We can observe that the resulting capital is greater when the games are switched more frequently. This behavior agrees with that of the original games [4]. Note that in Fig. 2 changing the value of ϵ affects only the vertical capital displacement; thus setting $\epsilon = 0$ pushes the graph into the positive region.

For the randomized games, we can now observe the volume of parameter space for which the paradox takes effect, by plotting the surfaces that represent the boundaries of the inequalities in Eq. (5). This is shown in Fig. 3, where for convenience we have set $p_2 = p_3$ to produce the graph in three variables. The volumes enclosed by the surfaces marked $Q_1, \ldots, Q_4$ are the regions of parameter space for which the paradox takes effect. Regions Q_1 and Q_3 are where two losing games combine to win. On the other hand, Q_2 and Q_4 represent the reverse effect where two winning games combine to lose. This conjugate region can be simply thought of in terms of changing the sign of the capital, so that the perspective of the concepts "win" and "lose" reverse. This was observed in the original capital-dependent games [27]; however, the conjugate regions were symmetrical. What is now interesting is that the new history-dependent games have asymmetrical conjugate regions, as can be seen in Fig. 3.

Another important comparison between the new history-dependent games and the original capital-dependent games is that the volume of parameter space is now bigger. A numerical mesh analysis on Fig. 3 revealed that the new games have a parameter space about 50 times larger than the original games reported in [27]. For applications such as in biophysics, it is important to find such gaming models

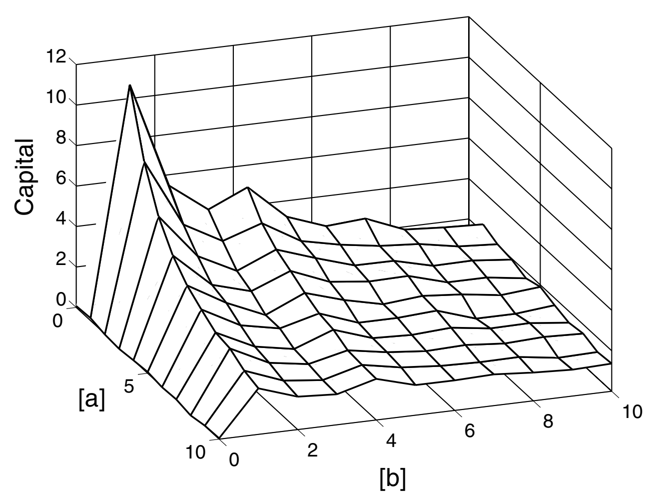

FIG. 2. Value of capital after 500 games. Games A and B′ are periodically mixed. Game A is played a times, followed by B′ played b times, and so on. Games are played with $\epsilon = 0$ and 500 000 ensemble averages have been taken. $p = 1/2$, $p_1 = 9/10$, $p_2 = p_3 = 1/4$, and $p_4 = 7/10$.

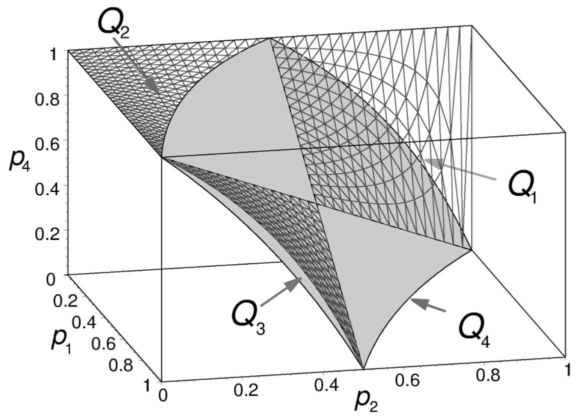

FIG. 3. Parameter space for game B′ when $p = 1/2$. We see there are four volumes labeled Q_1, Q_2, Q_3, and Q_4, bounded by the inequalities in Eq. (5). The paradox of two losing games that win if randomly combined occurs if the parameters lie within the volumes marked Q_1 and Q_3. In regions Q_2 and Q_4 the reverse effect occurs where games A and B are individually winning, but the randomized combination is losing.

with large and hence robust parameter spaces. Although it appears that the rates of winning from the slopes of Fig. 1 are about a factor of 2 lower than the original games, this is only the case for the particular chosen parameters. The 50 times increase in parameter space is favorable for applications in modeling evolutionary processes in biology, for example, where a weak payoff can gradually accumulate over a long period of time.

In summary, we have shown that the apparently paradoxical effect where two losing games can cooperate to win does work with a history-based state-dependent rule rather than the original restriction of a modulo capital-based state dependence. This, together with an increased parameter space, opens up the phenomenon to a wider range of possible application areas. This suggests that future investigation of further types of history-based rules and other types of state dependencies may be fruitful.

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[26] One could design a simpler non-Markovian game depending only on $X(t) - X(t-1)$, i.e., the outcome of the last run. However, this memory is too short to induce any paradoxical behavior. The reason is that in a fair game with such one-step memory the rules are always invariant under the change $X(t) \to -X(t)$, and this symmetry must be broken to have the paradox. A similar situation occurred in the original games where a modulo 3 rule was necessary since a modulo 2 rule is not enough to break the symmetry $X(t) \to -X(t)$.

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