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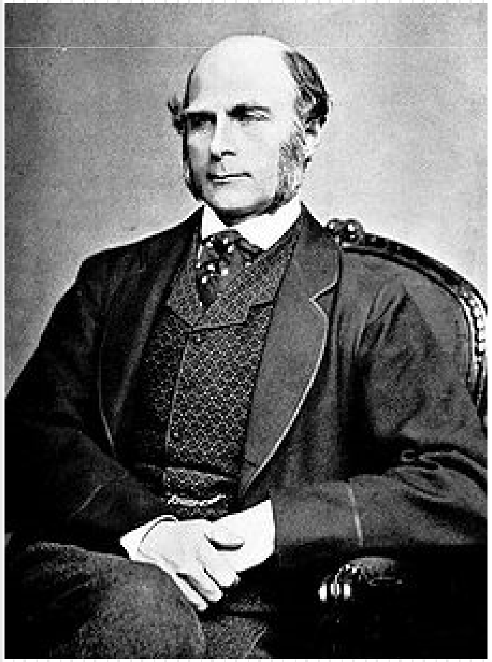
Branching Processes

Rev. H. William Watson



Sir Francis Galton

Summary

I. **Presentation of the branching processes**
- Introduction
- Markov chain
- Independent variable

II. **Main results and example**
- Generating function
- Expected value and variance
- Example

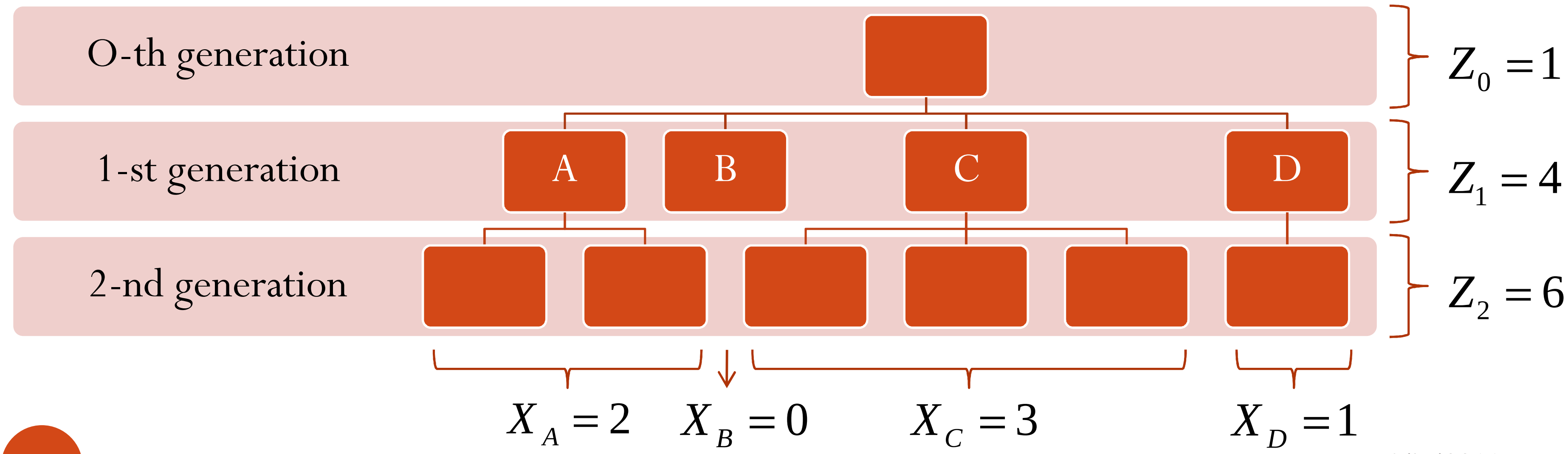
III. **Evolution of the system**
- Subcritical
- Critical
- Supercritical

IV. **Examples of branching processes**
- Family names
- Electron avalanche
- Neutron
- Other example

Summary

I Presentation of the branching processes

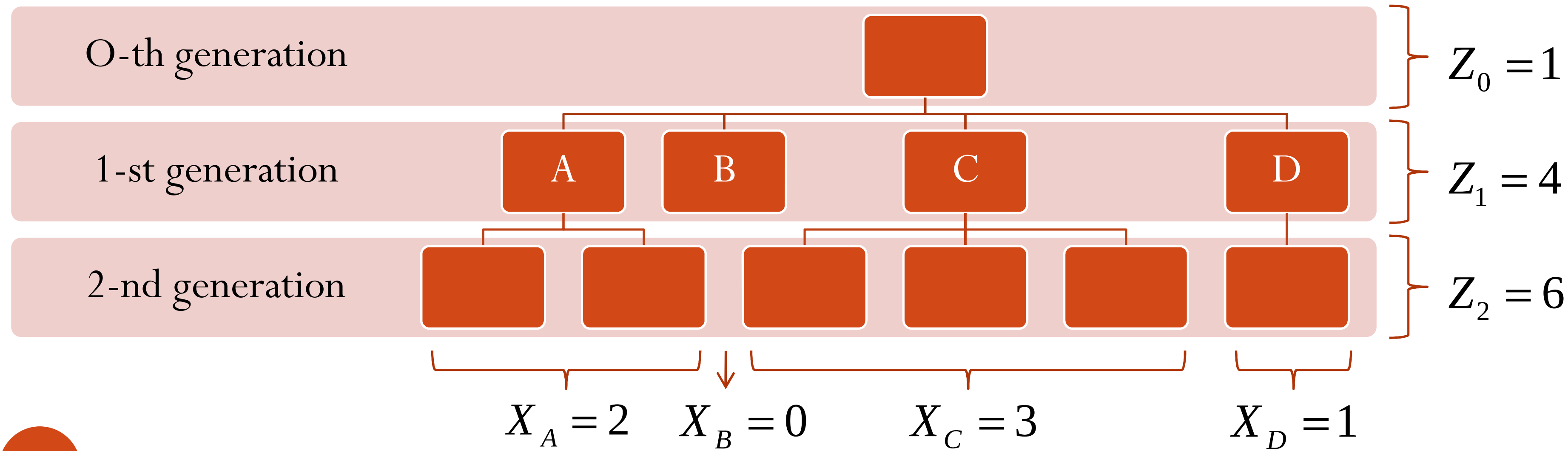
- An object can generate an object of the same kind.
- An initial object (the 0-th generation) has children (1-st generation); their children are called the 2-nd generation, etc…
- We keep track of the size of the succesives generation $Z_0, Z_1, Z_2...$ but not the time.



I Presentation of the branching processes

- We note $P(X_i = k) = p_k$ the probability to an object of the n-th generation to have k children in the existing in the (n+1)-th generation. With $k = 0, 1, 2, \ldots$ and $\sum p_k = 1$

- p_k don't depend on the generation number n.
- None of the probabilities $p_0, p_1 \ldots$ is equal to 1



I Presentation of the branching processes

Markov chain

- If the size of the n-th generation is known, probability governing the later generation is indenpendent of the sizes of the previous generations
- In other word $Z_0, Z_1, Z_2...$ form a Markov chain:

$$P_{ij} = P\left(Z_{n+1} = j \middle| Z_n = i\right) \qquad with \quad i, j, n = 0,1,2...$$

Independent variable

- The differents object do not interfer with one other. The number of children born is independent on how many other objects are presents
- If $Z_n = k$ then Z_{n+1} is distributed as a sum of k random variables:

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_i = \sum_{i=1}^{k} X_i$$

Summary

II Main results & example

Generating function:

$$f(s) = \langle s^X \rangle = \sum_{k=0}^{\infty} p_k s^k \qquad (1)$$

$$f_{n+1}(s) = \langle s^{Z_{n+1}} \rangle = \left\langle s^{\sum_{i=1}^{Z_n} X_i} \right\rangle = \langle s^{X_1} s^{X_2} \cdots s^{X_{Z_n}} \rangle = \left\langle \prod_{i=1}^{Z_n} s^{X_i} \right\rangle \qquad (2)$$

- Using (1) in (2) we obtain:

$$f_{n+1}(s) = \left\langle \prod_{i=1}^{Z_n} f(s) \right\rangle = \langle [f(s)]^{Z_n} \rangle = f_n(f(s))$$

$$f_{n+1}(s) = f_n(f(s))$$

II Main results & example

Expected value and variance

- The expected value and the variance of Z_1 are respectively given by:

$$m = E(Z_1) = \sum_{k=0}^{\infty} k \cdot p_k \qquad \sigma^2 = Var(Z_1) = E(Z_1^2) - m^2$$

- And we have for the expected value of Z_n :

$$E(Z_n) = m^n \qquad \text{with n= } 0,1,2,\ldots$$

- Concerning the variance of Z_n we have 2 formulas depending on the value of m:

$$Var(Z_n) = \begin{cases} \dfrac{\sigma^2 m^n (m^n - 1)}{m^2 - m} & if \quad m \neq 1 \\ n \cdot \sigma^2 & if \quad m = 1 \end{cases}$$

II Main results & example

Population of right whale in the the North Atlantic

- Generation length is one year
- A female right whale may produce 0, 1, or 2 females the following year
- the death of a parent results in the death of a calf in the first year.
- A female at time n produces:
  - no offspring if she dies before n + 1)
  - one offspring (herself) if she survives without reproducing female offspring
  - two offspring (herself and her calf) if she survives and gives birth to a female calf

II Main results & example

Population of right whale in the the North Atlantic

- p = Survival probability of the mother
- μ = Probability of begetting a female calf.

- The reproduction generating function of the process becomes:

$$f(s) = \langle s^X \rangle = \sum_{k=0}^{2} p_k s^k = (1-p)s^0 + p(1-\mu)s^1 + p\mu s^2$$

$$= 1 - p + p(1-\mu)s + p\mu s^2$$

- The mean value is:

$$m = p(1-\mu) + 2p\mu$$

II Main results & example

Population of right whale in the the North Atlantic

- The mean value is: $m = p(1-\mu) + 2p\mu$

- Using differents sources for μ and p, we obtain the following results for m:

	mu =0.051	mu =0.038
p=0.94	m=0.988	m= 0.976

- For initaly 150 females whale, and using other formula we can obtain:

m		0.988	0.976
Extinction with probability>0.99 within at most n years	n<	796	395

In the worst scenario the whale population will die out within 400 years with a probability of more than 99 %.

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IV Evolution of the system

- Since Z_n does not take the same positive value infinitely, it must go to ∞ or to 0, it doesn't remain positive and bounded.
  (The sequence $\{Z_n\}$ either go to ∞ or to 0).

- Writing the extinction probability q, we have:

$$Z_n \to 0 \quad with \quad a \quad probability \quad of \quad q = \frac{\#experiences\ in\ which\ Z_n \to 0}{\#total\ of\ experiences}$$

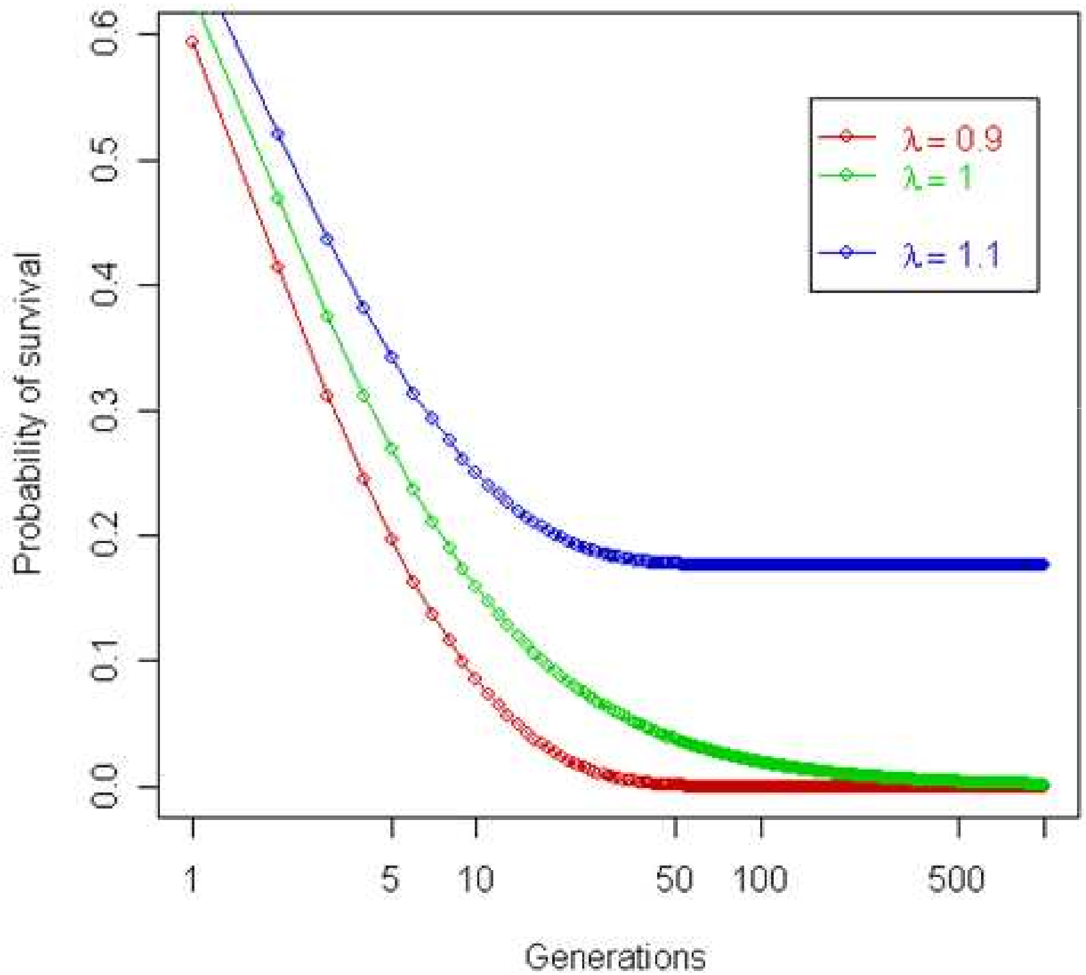
$$Z_n \to \infty \quad with \quad a \quad probability \quad of \quad 1-q$$

- Extinction $\Leftrightarrow$ the random sequence $\{Z_n\}$ consist of 0 except for a finite number of value of n $(Z_n \to 0)$

IV Evolution of the sytem

- **Subcritical** (*m<1*)
  - $q = 1.$
- **Critical** (*m =1*).
  - $q = 1$
- **Surpercritical** (*m>1*)
  - $q < 1$

  q is the unique positive solution of the equation $s = f(s)$ less than 1

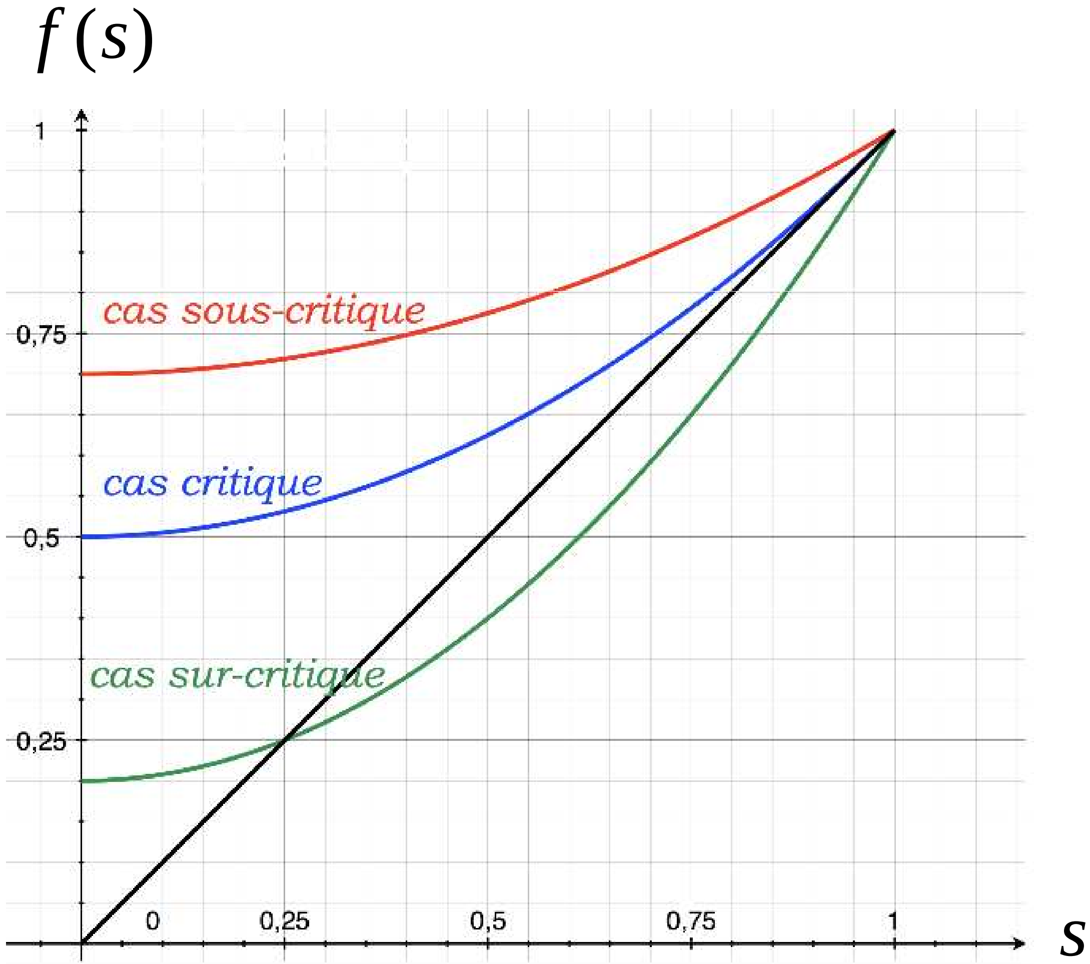


IV Evolution of the system

- **Subcritical (*m*<*1*)**
  - q = 1.
- **Critical (*m* =*1*).**
  - q = 1
- **Surpercritical (*m*>*1*)**
  - q < 1

  q is the unique positive solution of the equation $s=f(s)$ less than 1

Case in which $p_0 + p_2 = 1$

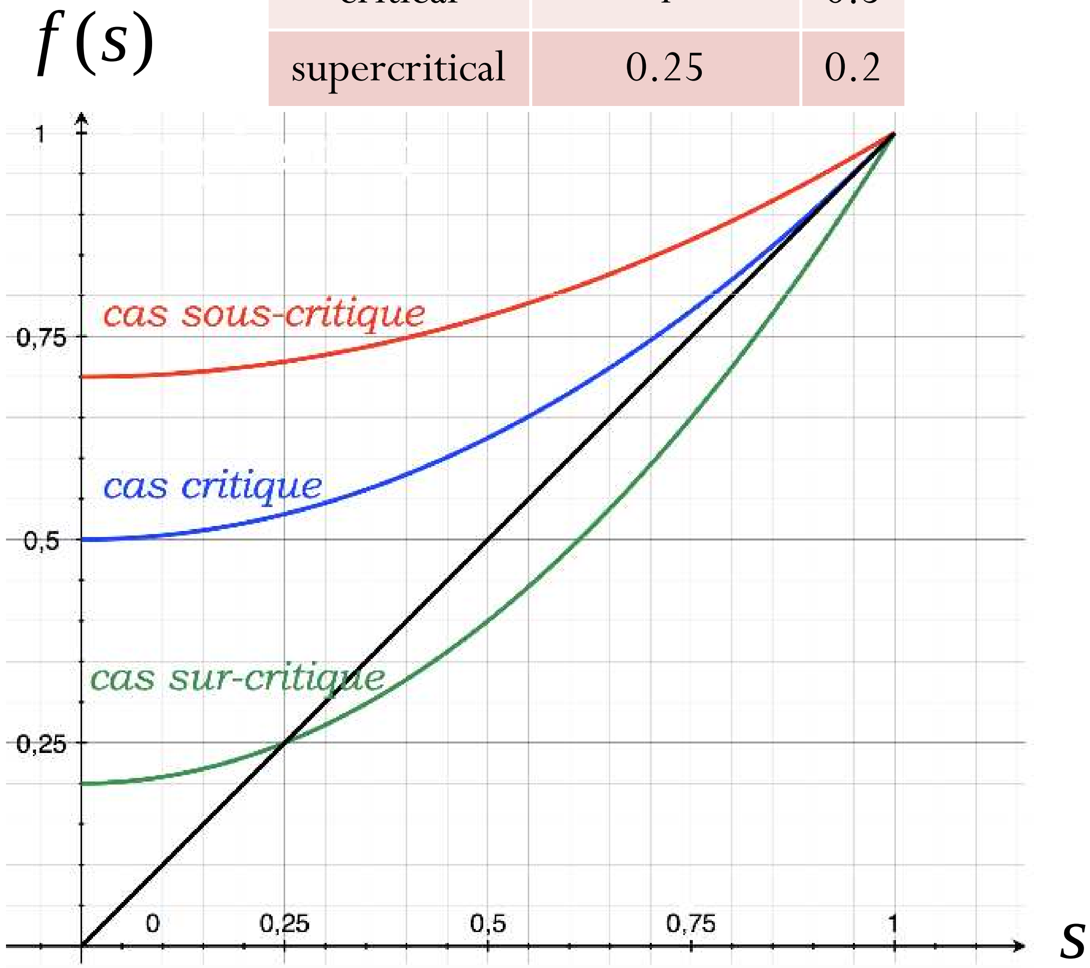


IV Evolution of the system

- **Subcritical** ***(m<1)***
  - ➢ q = 1.
- **Critical** ***(m =1).***
  - ➢ q = 1
- **Surpercritical** ***(m>1)***
  - ➢ q < 1

q is the unique positive solution of the equation $s=f(s)$ less than 1

regime	Extinction probability	Po
subcritical	1	0.7
critical	1	0.5
supercritical	0.25	0.2



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V Examples of branching processes

Original field of investigations: Family names

- There was concern amongst the Victorians that aristocratic families were becoming extinct.

- Francis Galton (1822-1911), anthropologist and polymath:
  - Are families of English peers more likely to die out than the families of ordinary men?

  - What is the probability that the male line goes extinct?

- Henry William Watson (1827-1903), vicar and mathematician gave the results.

V Examples of branching processes

Original field of investigations: Family names

- Due to an algebric error, Watson concluded wrongly that all families eventually die out.

- But Galton found a fact, that, with hindsight, provides a possible explanation for the observed data:
  - English peers tended to marry heiresses (daughters without brothers)
  - Heiresses come from families with lower fertility rates (lower probabilities $p_1, p_2, p_3, \ldots$).
  - . . . which increases the probability of the family dying out.

V Examples of branching processes

Original field of investigations: Family names

- Korean names are the most striking example, with only 250 family names, and 45% of the population sharing three family names

- China was the first country to use familly name (around 2852 before J.C.). Chinese names are similar, with 22% of the population sharing three family names (numbering close to 300 million people), and the top 200 names covering 96% of the population.

- Many Dutch names have included a family name since the Napoleonic Wars in the early 19th century, and there are over 68 000 Dutch family names.

V Examples of branching processes

2^{nd} field of investigation: Neutrons

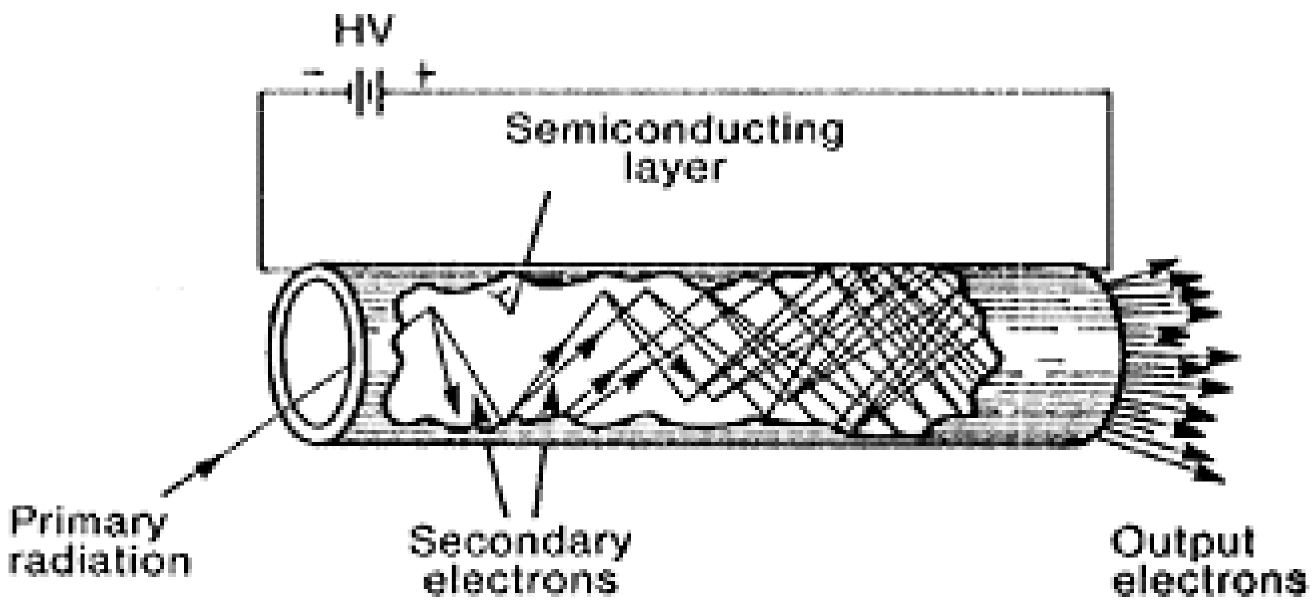
- Branching processes were reinvented by Leo Szilard in the late 1930s as models for the proliferation of free neutrons in a nuclear fission reaction.
- In nuclear fission devices are artificially operating at critical regime: m =1
- Generalizations of the extinction probability formulas played a role in the calculation of the critical mass of fissionable material needed for a sustained chain reaction.
- Szilard who convinced Einstein to suggest to President Roosevelt that the US begin the Manhattan project. (He won the Nobel Peace Prize in 1962)

V Examples of branching processes

Other field of applications: Electron avalanche

- In **electrons multiplier**:

  In order to amplify a weak current, each electron gives rise to a random number of electrons. Which strike the plate and produce more electrons, etc…



V Examples of branching processes

Other field of applications: Electron avalanche

In **ionisation detector**:

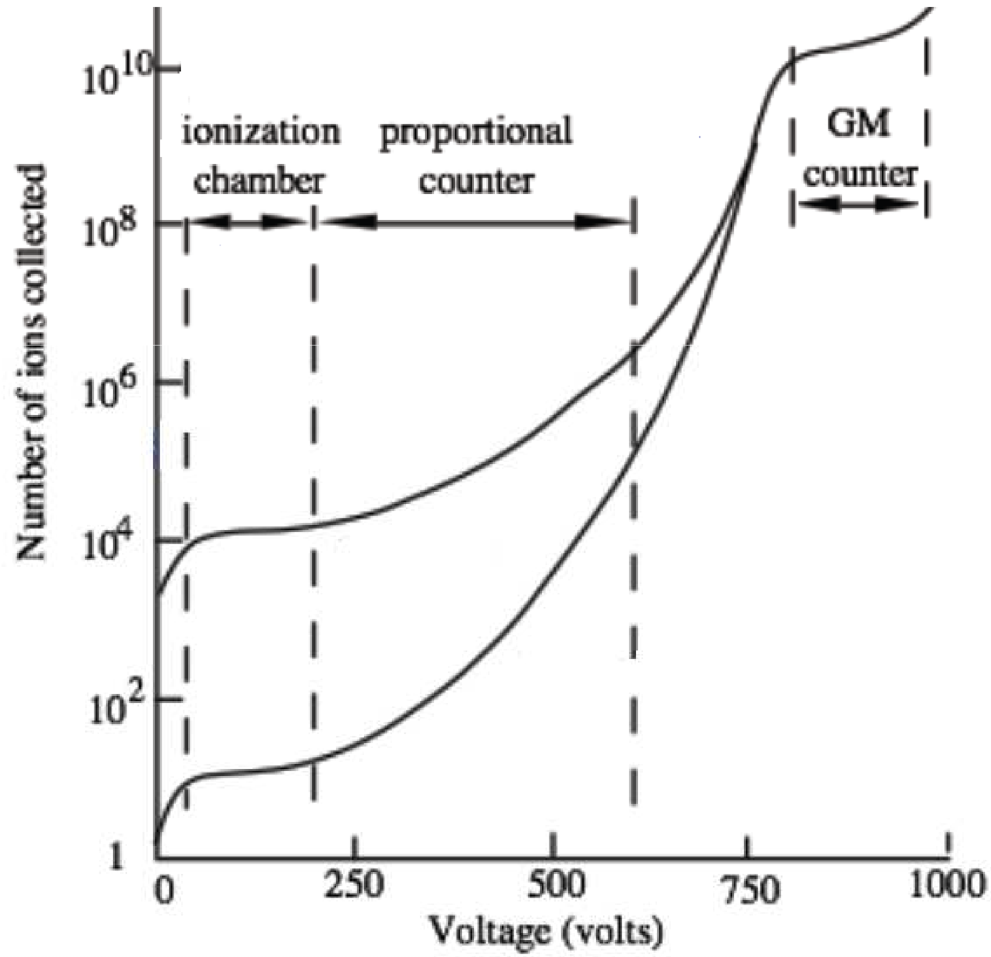
- **proportional counter**

  (non-Markovian)

  The single-electron pulse-height spectrum obtained in a measurements is not constant but depends upon the average final number of electrons in the avalanche.

- **Geiger-Muller detector**

  A Markov process based on the number of registered particles can be constructed



V Examples of branching processes

Other field of applications:

Reproduction of species

- For human (demography)
- For sickness (epidemiology)

In biology:

- Cell Cycle Model with Death and Quiescence
- Complexity Threshold in the Evolution of Early Life

Genetic

- Mutations in mitochondrial DNA
- Expansion of DNA Repeats

Cancer:

- Clonal Resistance Theory of Cancer Cells
- Drug Resistance and Chemotherapy

Others:

- Polymerase chain reaction
- Estimation of the Age of the Mitochondrial Eve

Thank you for your attention

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