From laser cooling to aging: A unified Lévy flight description

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What phenomena such as the subrecoil laser cooling of atoms and aging in glasses have in common is that these systems do not reach steady state during experimental observations, although the observation times are very large compared to microscopic time scales. We discuss some standard models that describe these phenomena and reformulate them in a unified framework in terms of the occupation times of the microscopic states of the system. A universal dynamical mechanism emerges, leading to a generic time-dependent distribution of occupation times, independent of the physical system considered. © 2008 American Association of Physics Teachers.

I. INTRODUCTION

One of the important paradigm shifts in 20th century physics was the role of probabilities, and hence fluctuations, as a central concept in the description of the physical world. This change arose through the development of both quantum mechanics and statistical mechanics. Many physical observables can be expressed as an average value plus fluctuations. Such a separation is useful when the fluctuations are small compared to the mean value, meaning that the probability distribution of the observable is narrow.

It has become clear since the work of Lévy in the 1930s1,2 that interesting situations arise when the probability distribution becomes so broad that the average value of the observable is not even defined (that is, it is formally infinite). This is the case when the tail of the probability distribution $p(x)$ behaves as a power law

$$p(x) \sim \frac{1}{x^{1+\alpha}}, \quad (x \to \infty),$$

with $0 < \alpha \leq 1$. In the early 1970s, it was realized that Lévy’s results were relevant to the study of random walks,3,4 with applications, for instance, to disordered conductors.5 In these systems some interesting and experimentally testable properties related to the underlying broad distributions arise, such as anomalous diffusion and aging. In the last 15 years broad distributions have found applications in physical processes as diverse as turbulent and chaotic transport,6–8 glassy dynamics,9–11 random walks in solutions of micelles,12 diffusion of spectral lines in disordered solids13,14 fluorescence of single nanocrystals,15 laser trapped ions,16,17 and laser cooling of atomic gases18,19 (for which the Nobel Prize was awarded in 1997).

The aim of this paper is to present an overview of some elementary models in which the dynamics involves a broad distribution of time intervals between successive events. We use simple probabilistic arguments to show that the equilibrium state cannot exist in a certain parameter range. We then describe the dynamics of the different models in terms of the mean time spent in a given microscopic state, and we denote this mean time as the occupation time of the state. A universal characterization of systems with nonstationary dynamics at large times emerges from this description. The common feature of these systems is the divergence of the average occupation time, which results from the presence of power-law tails in the distribution of the occupation times. We also calculate the probability of the system to be in a state with a given occupation time at time $t$ and find a universal form for this distribution at large times.

The paper is organized as follows. In Sec. II we give two examples of physical systems where the time spent by the system in different microscopic states becomes broadly distributed. In Sec. III we show that the two systems can be described by a unified framework, and calculate the corresponding dynamical distribution of the probability, before discussing some physical phenomena that can be explained by these results. Some details are given in Appendices A and B. In Appendix C we suggest a project appropriate for students.

II. SIMPLE PHYSICAL EXAMPLES

A. Laser cooling of atoms

Laser cooling of atomic gases consists of reducing the momentum spread of atoms due to momentum exchanges between atoms and photons. Subrecoil laser cooling consists of reducing the momentum spread of the atoms to less than a single photon momentum $\hbar k$, where $k = 2\pi/\lambda$ is the wave vector of the light, and $\hbar$ is the reduced Planck constant. This cooling is achieved by introducing a momentum dependence in the photon scattering rate in such a way that it decreases strongly in the vicinity of $p=0$, where $p$ is the magnitude of the atomic momentum $\mathbf{p}$. When an atom reaches by chance this region of momentum space, it tends to stay there for a long time, because the photon scattering rate is very low. Hence, atoms accumulate at small momenta, which amounts to a cooling of the system.18,19

The mechanism of subrecoil laser cooling is illustrated in Fig. 1. If a photon is absorbed and spontaneously reemitted by an atom, the atomic momentum undergoes a random jump of the order of $\hbar k$ because spontaneous emission occurs in a random direction. Thus, the repetition of absorption/spontaneous emission cycles generates a random walk of the momentum of the atom, with momentum-dependent time-in-
Fig. 1. Schematic picture of the subrecoil laser cooling process in momentum space. Photon scatterings induce a random walk for the momentum $p$ of the atoms, with steps of the order of $\hbar k$, the photon momentum. The mean waiting time $\tau(p)$ at momentum $p$ becomes very large for $p \approx p_m$, thus generating an accumulation of atoms near $p=0$.

tervals between two jumps. Due to this random motion, the atom may end up in the low momentum region, where it stays for a long time.

The time interval $\tau_w$ between two jumps is called the waiting time. It depends on the momentum of the atoms and fluctuates from one realization to another because photon scatterings occur randomly. An important quantity for characterizing the dynamics is the mean occupation time $\tau(p)$ (also called the waiting time) at momentum $p$, which is the average time spent between two successive photon scatterings. The occupation time $\tau(p)$ is the inverse of the photon scattering rate $R(p)$, so that the determination of the occupation time requires knowledge of the scattering rate. The photon scattering rate $R(p)$ vanishes for $p=0$. In most cases, the small $p$ behavior of $R(p)$ is given by

$$R(p) = \frac{1}{\tau_0}(\frac{p}{p_m})^\beta, \quad (p \to 0),$$

(2)

where $\tau_0$ is a time scale, and $p_m$ is a momentum scale to be specified. The exponent $\beta$ is an even integer, $\beta=2$, 4, or 6, which depends on the specific atom-light interaction process.\(^{18,19}\) It follows that $\tau(p)$ behaves as small $p$ as a (diverging) power law,

$$\tau(p) = \tau_0\left(\frac{p_m}{p}\right)^\beta, \quad (p \to 0).$$

(3)

Hence, for small enough $p$, the time spent at momentum $p$ becomes very large with respect to the time spent at large momentum. To be specific, we can find a momentum scale $p_m \approx \hbar k$ (where $\hbar k$ is the typical change in the atomic momentum when a photon is scattered) such that $R(p)$ behaves as a power law for $p < p_m$, and such that the time spent by the atom in the region $p > p_m$ is very small compared to the time spent at $p < p_m$. Note that as $p_m \approx \hbar k$, values of $p$ smaller than $p_m$ are reached uniformly after a photon scattering from a larger momentum $p = \hbar k$.

To model subrecoil laser cooling in a simplified way, we neglect the time spent outside the region $p < p_m$, and assume that an atom leaving this low momentum region instantaneously returns to it. We choose at random a new momentum within a sphere of radius $p_m$, with a uniform distribution in $d$ dimensions. Thus, the distribution $\rho(p)$ of the momentum $p$ reached after a jump into the low-momentum region is

$$\rho(p) = \frac{d}{p_m^d}p^{d-1}, \quad (0 < p < p_m).$$

(4)

The absence of memory between two successive values of the momentum means that the dynamics corresponds to a Markov process (see Sec. III A).

We now introduce another probability distribution, the probability $P_a(p)$ for an atom to have the momentum $p$ after a steady state is reached. This probability is proportional to the occupation time $\tau(p)$ times the probability $\rho(p)$ of reaching a momentum $p$ after a jump. Hence, $P_a(p)$ is

$$P_a(p) = \frac{1}{\langle \tau(p) \rangle}\rho(p) = \frac{\tau_0 p_m^\beta}{\langle \tau(p) \rangle}p^{d-\beta-1}\rho(p),$$

(5)

where the average occupation time $\langle \tau(p) \rangle$, computed over the distribution $\rho(p)$ is given by

$$\langle \tau(p) \rangle = \int_0^{p_m} dp\, \rho(p) \tau(p) = \frac{\tau_0 p_m^\beta}{\langle \tau(p) \rangle}p^{d-\beta-1}.$$
\[ \rho(V) = V_0^{-1}e^{-V/V_0}, \quad (V > 0), \]  
(7)
where the quantity \( V_0 \) is an energy scale. In the following, we will consider the limiting case of an infinite number of traps.

The dynamics of the model is defined by the distribution \( \varphi(\tau_0|V) \) of the waiting times \( \tau_0 \) in a trap of depth \( V \). Here we consider a Markov process, which means that the waiting times associated with successively visited traps are uncorrelated. For Markov processes the distribution of waiting times is exponential, as discussed in Sec. III A. We thus consider the following exponential form for \( \varphi(\tau_0|V) \), namely

\[ \varphi(\tau_0|V) = \tau(V)^{-1}e^{-\tau_0/\tau(V)}, \]  
(8)
where \( \tau(V) \) is the mean waiting time, that is, the occupation time of the trap. The probability per unit time to escape the trap is constant in a given trap and is equal to \( 1/\tau(V) \).

From thermal activation arguments the occupation time \( \tau(V) \) of a trap of depth \( V \) at temperature \( T \) is given by the Arrhenius form,

\[ \tau(V) = \tau_0 e^{V/k_BT}, \]  
(9)
where \( \tau_0 \) is the microscopic time scale that characterizes the random motion of the particle within the trap, and \( k_B \) is Boltzmann’s constant.

In equilibrium the probability to be in a trap of depth \( V \) is proportional to the product of the trap occupation time and the density distribution \( \rho(V) \) of traps of depth \( V \),

\[ P_{eq}(V) = \frac{1}{\langle \tau(V) \rangle} \rho(V) \tau(V) = \frac{\tau_0}{\langle \tau(V) \rangle} \rho(V) e^{V/k_BT}, \]  
(10)
which is the usual Boltzmann–Gibbs distribution. Note the + sign in the exponential, because \( V \) is the opposite of the energy \( U \) of the trap. The mean occupation time \( \langle \tau(V) \rangle \), averaged over all traps, is given by

\[ \langle \tau(V) \rangle = \tau_0 \int_0^\infty dV \rho(V) e^{V/k_BT}. \]  
(11)
Using Eq. (7), we find

\[ \langle \tau(V) \rangle = \frac{\tau_0}{k_BT_0} \int_0^\infty dV \exp \left[ \frac{1}{T - \tau_0/k_BT} \right] \frac{V}{T}. \]  
(12)
where the temperature \( T_0 = V_0/k_BT \) has been introduced. If \( T \approx T_0 \), the integral in Eq. (12) diverges, and \( \langle \tau(V) \rangle \) is infinite. Accordingly, the distribution defined in Eq. (10) is no longer normalizable. Thus, the divergence of the average occupation time rules out the existence of an equilibrium state and leads to a nonstationary dynamics at long times. If the system starts from a uniform distribution among the traps (which can be interpreted as an infinite temperature initial condition), the average energy of the system drifts slowly toward lower values without ever reaching a steady state.

III. UNIFIED DESCRIPTION IN TERMS OF OCCUPATION TIMES

A. Distribution of occupation times

We have seen that broad distributions of occupation times can be found in different physical systems. These problems are usually studied using the natural physical variables, such as momentum for laser cooling\(^{19}\) and the energy barrier in the trap model.\(^ {23,24}\) These problems become formally equivalent if expressed in terms of occupation times, at least when considering the dynamical distribution of occupation times. Differences only appear when we look at the predictions for the physical variables, as discussed in Sec. III D.

Let us calculate the distribution of occupation times \( \psi(\tau) \) for the laser cooling and the trap models. If we start from Eqs. (3) and (4) and Eqs. (7) and (9) and use the relation \( \rho(p)|dp| = \psi(\tau)|d\tau| \) and \( \rho(E)|dE| = \psi(\tau)|d\tau| \), we find a power law behavior for \( \psi(\tau) \) in both models,

\[ \psi(\tau) = A\tau^{-\alpha}, \quad (\tau > \tau_{\text{min}}), \]  
(13)
with \( \alpha = d/\beta \) for laser cooling and \( \alpha = T/T_0 \) for the trap model; \( A \) and \( \tau_{\text{min}} \) are model-dependent parameters, and their values are unimportant in the following. For \( \alpha < 1 \) the first moment \( \langle \tau \rangle \) diverges, which means that the occupation time averaged over all possible states is infinite.

At this stage, we emphasize that two different notions of time intervals appear in these problems, namely the waiting time \( \tau_w \) and the occupation time \( \tau \). The waiting time is the time spent between two successive events—scatterings by a photon for laser cooling, and escapes from a trap in glassy dynamics. The occupation time is associated with a microscopic state of the system, and is defined as the average value of the waiting time in this state. Occupation times can also be used to label the corresponding states, instead of using the natural physical variables of the system. In addition, the models we consider are Markov processes, which means that for a given occupation time \( \tau \), the probability of exiting the current state during a short time interval \( dt \) is \( dt/\tau \). As a result, the distribution \( \varphi(\tau_0|\tau) \) of waiting times \( \tau_w \) in a state with occupation time \( \tau \) is exponential:

\[ \varphi(\tau_0|\tau) = \frac{1}{\tau} e^{-\tau_0/\tau}. \]  
(14)
The full distribution \( \psi_w(\tau_w) \) of waiting times, averaged over all microscopic states, is

\[ \psi_w(\tau_w) = \int_0^\infty d\tau \varphi(\tau_0|\tau) \psi(\tau) = \int_0^\infty \frac{d\tau}{\tau} \psi(\tau) e^{-\tau_0/\tau}. \]  
(15)
If \( \psi(\tau) \) has a power-law tail for \( \tau \to \infty \), then \( \psi_w(\tau_w) \) also has a power-law tail with the same exponent as \( \psi(\tau) \).

B. Master equation for the dynamical distribution of occupation times

To obtain a unified picture of the models discussed in Sec. II, we characterize the states of the system by their occupation times \( \tau \) without any reference to the physical meaning of these states. We wish to determine the probability density \( P(\tau,t) \) to be in a state with occupation time \( \tau \) at time \( t \). For a Markov process, the evolution of \( P(\tau,t) \) is given by the master equation (see Appendix A):

\[ \frac{\partial P(\tau,t)}{\partial t} = -P(\tau,t) \int_0^\infty d\tau' W(\tau \to \tau') \]
\[ + \int_0^\infty d\tau' W(\tau' \to \tau) P(\tau',t) , \]  
(16)
where \( W(\tau \to \tau') \) is the transition rate from states with occu-
pation time $\tau$ to states with occupation time $\tau'$. The master equation is a balance equation for the probability $P(\tau,t)$, taking into account the probability to leave states with occupation time $\tau$, and the probability to reach them starting from another state. We make the important assumption that the occupation times $\tau_1, \tau_2, \ldots, \tau_n$ of the states successively occupied by the system are statistically independent. Hence, at each transition, a new occupation time $\tau'$ is chosen at random according to the distribution $\psi(\tau')$. Given that the probability per unit time to leave state $\tau$ is $1/\tau$, the transition rate thus takes the form

$$W(\tau \to \tau') = \frac{1}{\tau} \psi(\tau').$$

(17)

The lack of correlation between the states before and after a jump occurs physically because of the strongly stochastic underlying dynamics. In subrecoil laser cooling it comes from the fact that in between two trapping events, the atoms perform a random walk outside the trap, which takes a negligible time, but decorrelates two consecutive trapping moments. A similar scenario holds for aging dynamics, where the system wanders among high energy states and fully decorrelates before finding a new trap.

Note that the independence property, which is stronger than the Markovian property, would lead to a trivial stationary process if all states had the same occupation time. As we will show, interesting nonstationary effects appear when the occupation times are broadly distributed.

We use the form of the transition rates given in Eq. (17) and the normalization of $\psi(\tau')$ to write the master equation Eq. (16) as

$$\frac{\partial P(\tau,t)}{\partial t} = -\frac{1}{\tau} P(\tau,t) + \psi(\tau) \int_0^\infty \frac{d\tau'}{\tau} P(\tau',t).$$

(18)

C. Solution of the master equation

The master equation [Eq. (18)] can be solved exactly using a Laplace transform with respect to $t$. The calculations are described in detail in Appendix B, and we summarize the main results in the following. The Laplace transform $\hat{P}(\tau,s)$ of $P(\tau,t)$ is defined as

$$\hat{P}(\tau,s) = \int_0^\infty dt e^{-st} P(\tau,t).$$

(19)

If we assume that at time $t=0$ the system is distributed at random over all possible states, then the initial distribution $P(\tau,t=0)$ reduces to $\psi(\tau)$. In this case $\hat{P}(\tau,s)$ can be written in the simple form

$$\hat{P}(\tau,s) = \frac{1}{s} \frac{\tau \psi(\tau)}{s^* (s) 1 + s \tau},$$

(20)

where the time scale $s^*(s)$, to be interpreted as an effective average occupation time, is defined as

$$s^*(s) = \int_0^\infty d\tau \frac{\tau \psi(\tau)}{1 + s \tau}.$$

(21)

Note that $\hat{P}(\tau,s)$ is normalized to $1/s$ with respect to $\tau$. Equation (20) is an exact result that does not rely on the long time regime $(s \ll s_0^{-1})$ approximation. The influence on the dynamics of the finiteness or divergence of the average occupation time $\langle \tau \rangle$ now appears clearly, because $\langle \tau \rangle = \lim_{s \to 0} s^*(s)$. If $\langle \tau \rangle$ is finite, the distribution $\hat{P}(\tau,s)$ converges for $s \to 0$ (corresponding to $t \to \infty$) to the equilibrium distribution (up to the usual $1/s$ factor)

$$\hat{P}(\tau,s) \approx \frac{1}{s \langle \tau \rangle} \psi(\tau), \quad (s \to 0).$$

(22)

Thus, in this case the system reaches an equilibrium state at infinite time. The convergence to this asymptotic state may be very slow, typically as a power law with an exponent that becomes small when $\langle \tau \rangle$ is large.

In contrast, if $\langle \tau \rangle$ is infinite, the effective average occupation time $\langle \tau^* \rangle(s)$ grows without bound as $s \to 0$, and no stationary state is ever reached. If the occupation time distribution $\psi(\tau)$ has a power-law tail,

$$\psi(\tau) \sim \frac{c}{\tau^{1+\alpha}}, \quad (\tau \to \infty),$$

(23)

with $0 < \alpha < 1$, then the out-of-equilibrium solution of the master equation exhibits some interesting properties, in particular a scaling form at long times $(s \to 0)$. To be more specific, the distribution $\hat{P}(\tau,s)$ can be written asymptotically as

$$\hat{P}(\tau,s) \approx \tilde{\psi}(s \tau), \quad (s \to 0).$$

(24)

The scaling function $\tilde{\psi}(x)$ can be determined from Eq. (20), namely,

$$\tilde{\psi}(x) = \frac{\sin(\pi \alpha)}{\pi} \frac{1}{x^\alpha (1 + x^\alpha)}.$$

(25)

The inverse Laplace transform of $\hat{P}(\tau,s)$ in the scaling regime can be calculated in terms of physical time $t$. As expected, the resulting distribution $P(\tau,t)$ also takes a scaling form,

$$P(\tau,t) = \frac{1}{t} \tilde{\psi} \left( \frac{\tau}{t} \right),$$

(26)

which shows that the typical occupation time $\tau_t$ at time $t$ is of the order of $t$. [Note that $\psi(u)$ is not the inverse Laplace transform of $\tilde{\psi}(x)$.] The scaling function $\psi(u)$ is such that $\psi(u) \sim u^{-\alpha}$ for $u \to 0$ and $\psi(u) \sim u^{1-\alpha}$ for $u \to \infty$. The asymptotic behavior of $P(\tau,t)$ for $\tau \ll t$ or $\tau \gg t$ can then be calculated, and is found to be qualitatively the same as that of $\hat{P}(\tau,s=1/t)$. If we neglect multiplicative constants, the distribution $P(\tau,t)$ takes for $\tau \ll t$ the pseudoequilibrium form

$$P(\tau,t) \sim \tau \psi(\tau), \quad (\tau \ll t).$$

(27)

For $\tau \gg t$, $P(\tau,t)$ remains proportional to the a priori distribution $\psi(\tau)$,

$$P(\tau,t) \sim \psi(\tau), \quad (\tau \gg t).$$

(28)

The interpretation of Eq. (28) is that states with small enough occupation times have been visited a large number of times and are equilibrated. In contrast, states with very large occupation times have been visited only once, and the precise
value of their occupation time does not affect the dynamics at this stage.

D. Link between occupation time and physical observables

Even though the different models introduced in Sec. II share a common dynamical distribution $P(\tau, t)$, the physical observables do not behave in the same way in these models. The reason is that the functional relations between these observables and the occupation times differ.

In the subcooler laser cooling problem, the relation between the momentum $p$ and the occupation time $\tau$ is $p = p_{\text{in}}(\tau/\tau_0)^{-1/\beta}$. Hence, from Eq. (26) the dynamical distribution of momentum $P_m(p, t)$ can be written in the long time regime as

$$P_m(p, t) = \frac{\beta p p_{\text{in}}^\beta}{tp^{1+\beta}} \phi \left( \frac{p p_{\text{in}}^\beta}{tp^{1+\beta}} \right).$$

Thus, the distribution of momentum becomes narrower, and the typical momentum $p_t$, reached after a long time $t$ is given by

$$p_t \approx p_{\text{in}}(t/\tau_0)^{-1/\beta}.$$  \hspace{1cm} (30)

Hence, the atoms can in principle be cooled to arbitrarily low temperatures if we wait long enough and no other effects are present. This result is in remarkable agreement with the observations made in laser cooling experiments,\(^{19}\) and leads to a clear understanding of this phenomenon, which is of purely dynamical origin.

In the trap model of glass aging, the energy $U$ is related to the occupation time $\tau$ by $U = -T \ln(\tau/\tau_0)$. Because the typical occupation time at time $t$ is of the order of $t$, the average energy $U^a(t)$ behaves as

$$U^a(t) \approx - T \ln(t/\tau_0).$$  \hspace{1cm} (31)

The logarithmic behavior of the energy is frequently observed in aging systems, and is accounted for in a simple way in the present approach. The latter gives a precise meaning to aging, because it is clear from the scaling form in Eq. (26) and from the average energy given in Eq. (31) that the system keeps track of its age.

IV. CONCLUSION

We have used two examples to illustrate how broad distributions can lead to deep insights and to a unified view on the physics of systems that do not reach equilibrium. The formalism leads in a straightforward way to the longtime behavior of such systems, and shows that this behavior is universal, depending only on the exponent $\alpha$ of the power-law tail of the occupation time distribution. In addition, this approach gives a well-defined meaning to the notion of quasi-equilibrium at long times. That is, microscopic states with occupation times less than $t$ are essentially equilibrated, whereas those with an occupation time greater than $t$ are not. Finally, we recall that the return time between two scattering or trapping events was neglected in the present approach. More general situations, for which the return time is not negligible, may also be considered. Such generalizations do not change the main qualitative conclusions reported here.\(^{19}\)

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APPENDIX A: MASTER EQUATION FORMALISM

We consider a system described by a finite set of configurations (or microscopic states) $|C\rangle$. The set $|C\rangle$ is the configuration space of the system. We assume that the system evolves according to a Markovian stochastic dynamics, which means that the system changes configurations randomly in time without memory of the configurations occupied earlier. To be specific, we assume that the system is in an arbitrary configuration $C$ at time $t$. The probability that the system goes from $C$ to a new configuration $C'$ between time $t$ and $t+dt$, where $dt$ is a very short time interval, is equal to $W(C \rightarrow C')dt$ (the probability per unit time) from $C$ to $C'$. Roughly speaking, the Markovian property of the dynamics means that $W(C \rightarrow C')$ depends neither on the history of the system before arriving at configuration $C$, nor on the time spent in configuration $C$. The quantity $W(C \rightarrow C')$ should be understood as the probability per unit time to go to configuration $C'$ given that the system is in configuration $C$; it is thus a conditional probability.

We now introduce the probability $P_C(t)$ for the system to be in a configuration $C$ at time $t$. The evolution with time of $P_C(t)$ is described by a master equation, which we obtain as follows. In the time interval $dt$, the variation $dP_C$ of $P_C(t)$ is due to the probability $d\Phi^\text{out}_{C}$ of leaving the configuration $C$ during the time interval $dt$, and the probability $d\Phi^\text{in}_{C}$ of reaching the configuration $C$ from any other configuration $C'$ in the same time interval. We thus have

$$dP_C = -d\Phi^\text{out}_{C} + d\Phi^\text{in}_{C}. \hspace{1cm} (A1)$$

These two contributions can be evaluated as follows. Starting from $C$, the system may go to any other configuration $C'$, so that the corresponding probability is the sum of all such possible transition probabilities. The probability to be in configuration $C$ and to go to configuration $C'$ in time $dt$ is $W(C \rightarrow C')P_C(t)dt$, which yields

$$d\Phi^\text{out}_{C} = \sum_{C' \neq C} W(C \rightarrow C')P_C(t)dt. \hspace{1cm} (A2)$$

Similarly, $d\Phi^\text{in}_{C}$ is obtained by summing the transition probabilities from all other configurations $C'$ to configuration $C$,

$$d\Phi^\text{in}_{C} = \sum_{C' \neq C} W(C' \rightarrow C)P_{C'}(t)dt. \hspace{1cm} (A3)$$

If we substitute Eqs. (A2) and (A3) into Eq. (A1), we obtain

$$\frac{dP_C}{dt} = -\sum_{C' \neq C} W(C \rightarrow C')P_C(t) + \sum_{C' \neq C} W(C' \rightarrow C)P_{C'}(t),$$

which is the master equation describing the stochastic evolution of the system. For simplicity, our argument was restricted to systems with a finite number of discrete configu-
APPENDIX B: EXPLICIT SOLUTION OF THE MASTER EQUATION

We give here the exact solution of the master equation given in Eq. (18). We introduce \( \hat{P}(\tau, s) \) defined in Eq. (19) and write the master equation as:

\[
s \hat{P}(\tau, s) - P_0(\tau) = - \frac{1}{s} \hat{P}(\tau, s) + \psi(\tau) \int_0^\infty \frac{d\tau'}{\tau} \hat{P}(\tau', s),
\]

where \( P_0(\tau) \) is the initial condition, \( P_0(\tau) = P(\tau, t=0) \). The left-hand side in Eq. (B1) is obtained by an integration by parts. We solve for \( \hat{P}(\tau, s) \) and find

\[
\hat{P}(\tau, s) = \frac{\tau P_0(\tau)}{1 + s \tau} + \frac{\tau \psi(\tau)}{1 + s \tau} \phi(s),
\]

with \( \phi(s) = \int_0^\infty d\tau' \tau'^{-1} \hat{P}(\tau', s) \). We first solve for \( \phi(s) \). If we divide Eq. (B2) by \( \tau \) and integrate over \( \tau \), we have

\[
\int_0^\infty \frac{d\tau}{\tau} \hat{P}(\tau, s) = \int_0^\infty d\tau' \frac{P_0(\tau)}{1 + s \tau} + \phi(s) \int_0^\infty d\tau' \frac{\psi(\tau)}{1 + s \tau}.
\]

The solution for \( \phi(s) \) is

\[
\phi(s) = \frac{1}{s} \left( \int_0^\infty d\tau' \frac{\tau \psi(\tau')}{1 + s \tau'} \right)^{-1} \int_0^\infty d\tau' \frac{P_0(\tau)}{1 + s \tau'}.
\]

We substitute this solution in Eq. (B2) and obtain

\[
\hat{P}(\tau, s) = \frac{\tau P_0(\tau)}{1 + s \tau} + \frac{1}{s} \frac{\tau \psi(\tau)}{1 + s \tau} \left( \int_0^\infty d\tau' \frac{\tau' \psi(\tau')}{1 + s \tau'} \right)^{-1} \int_0^\infty d\tau' \frac{P_0(\tau')}{1 + s \tau'}.
\]

If we use the time scale \( \tau_0^*(s) \) introduced in Eq. (21), the solution of the master equation can be written as

\[
\hat{P}(\tau, s) = \frac{\tau P_0(\tau)}{1 + s \tau} + \frac{1}{s} \frac{\tau \psi(\tau)}{1 + s \tau_0^*(s)} \int_0^\infty d\tau' \frac{P_0(\tau')}{1 + s \tau'}.
\]

The quantity \( \tau_0^*(s) \) may be interpreted as an effective average occupation time, in the sense that it is an average of \( \tau \) over the distribution \( \psi(\tau) \), taking into account a cut-off of dynamical origin around the time scale \( \tau^* = s^{-1} \). For the case where \( P_0(\tau) = \psi(\tau) \), Eq. (B6) simplifies to Eq. (20) after a straightforward calculation.

It is interesting to introduce a second time scale \( \tau_0^*(s) \) associated with \( P_0(\tau) \) by

\[
\tau_0^*(s) = \int_0^\infty d\tau' \frac{\tau P_0(\tau)}{1 + s \tau'}.
\]

We can then rewrite Eq. (B6) in the more compact form

\[
\hat{P}(\tau, s) = \frac{1}{s \tau_0^*(s)} \frac{\tau}{1 + s \tau} \left[ s \tau_0^*(s) P_0(\tau) + (1 - s \tau_0^*(s)) \psi(\tau) \right].
\]

The time scale \( \tau_0^*(s) \) is the effective average time associated with the initial distribution \( P_0(\tau) \), taking into account the cut-off time scale \( \tau^* = s^{-1} \).

In the long time limit only the asymptotic behavior of \( \hat{P}(\tau, s) \) for \( s \to 0 \) is relevant. In this limit the expression in square brackets in Eq. (B8) reduces to \( \psi(\tau) \), because \( s \tau_0^*(s) \) and \( s \tau_0^*(s) \) go to zero (even though \( \tau_0^*(s) \) and \( \tau_0^*(s) \) may diverge when \( s \to 0 \), and we recover the same result as Eq. (20); the memory of the initial condition is lost. In particular, the scaling form in Eq. (24) and the functional form of the scaling function \( \phi(s) \) given in Eq. (25) do not depend on the initial condition. The crossover time \( t = s^{-1} \) beyond which the scaling form Eq. (24) becomes valid depends on \( P_0(\tau) \). For instance, if \( P_0(\tau) \) has a peak around \( \tau_p \gg \tau_0 \), the scaling regime is reached only for times \( t \gg \tau_p \), so that it may be necessary to wait for an arbitrarily large time.

The inverse Laplace transform of Eq. (24) leads to the scaling form Eq. (26), where the scaling function \( \phi(u) \) is given by

\[
\phi(u) = \frac{\sin \pi \alpha u}{\pi \Gamma(\alpha)} e^{-1/u} \int_0^1 du v^{\alpha-1} e^v.
\]

An asymptotic expansion of Eq. (B9) for large and small values of \( u \), respectively, leads to the asymptotic behavior of \( P(\tau, t) \) given in Eqs. (27) and (28).

APPENDIX C: SUGGESTED PROJECT

The dynamical distribution \( P(\tau, t) \) and the scaling behavior given in Eq. (26) can be obtained using a simple numerical simulation of a random process. Draw a random number \( u \) from a uniform distribution between 0 and 1 and calculate the occupation time \( \tau \) of the current state as \( \tau = \tau_0 e^{-1/\alpha} \) (the microscopic time \( \tau_0 \) may be set to unity). Check that the probability distribution \( p(\tau) \) satisfies

\[
p(\tau) = \frac{\alpha \tau_0^\alpha}{\tau^{1+\alpha}}, \quad (\tau > \tau_0).
\]

This calculation can be done either numerically or analytically noticing that \( p(\tau) d\tau = |du| \). Once a state with occupation time \( \tau \) is given, the waiting time \( \tau_w \) in this state is chosen randomly from an exponential distribution of mean \( \tau \). To do so, draw a uniform random number \( u, 0 < u < 1 \), and compute \( \tau_w = -\tau \ln u \). Check again that \( \tau_w \) has the desired distribution. Then iterate this calculation and sum the different waiting times until the final time \( t \) is reached (\( t \) is an arbitrary fixed time such that \( t > T_0 \), typically in the range \( t = 10^3 \tau_0 \) to \( 10^6 \tau_0 \), and record the occupation time \( \tau \) of the state occupied at time \( t \).

Repeat this procedure many times, and compute the histogram of the values of \( \tau \) at time \( t \). Normalizing the histogram yields the distribution \( P(\tau, t) \). Compute this distribution for different times, and check that for \( \alpha > 1 \), it becomes independent of \( t \) in the long time limit; for \( \alpha < 1 \) the \( \tau \)-dependence re-
mains for arbitrarily large $t$. In the latter case determine the scaling form Eq. (26) and the asymptotic behavior of $P(\tau,t)$ for $\tau \ll t$ and $\tau \gg t$.

Alternatively, these simulations may be performed by considering the momentum $p$ of the laser cooling model or the energy barrier $V$ of the trap model instead of the occupation time $\tau$.

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Deceased.